Solution to homework for Session 6 August 2008

Problem:

Consider the following definitions for a dyadic deontic operator. Again, I is a set of propositional formulas meant to represent the "termination statements" of a set of unconditional imperatives (the indicative statements that are true iff the imperative is satisfied), and $I\perp\neg C$ is the set of $\neg C$-remainders of I (the set of all maximal subsets Γ of I such that $\Gamma \nvdash_{PL} \neg C$):

(a) $I \vDash O(A/C)$ iff $\forall\, \Gamma \in I\perp\neg C$: $\Gamma \cup \{C\} \vdash_{PL} A$

(b) $I \vDash O(A/C)$ iff $\exists\, \Gamma \in I\perp\neg C$: $\Gamma \cup \{C\} \vdash_{PL} A$

Let $DDL^{\forall}$ be the axiomatic system that corresponds to the semantics that employs definition (a), and $DDL^{\exists}$ the system that corresponds to definition (b) (for the description of the systems cf. the handout for the session from 6 August).

[*Note:* Since (a) defines a 'sceptic' nonmonotonic operator, I have elsewhere used the name DDL^{S} for $DDL^{\forall}$, and since its 'bold' counterpart (b) was invented by Bas van Fraassen (cf. B. van Fraassen. "Values and the Heart's Command." *Journal of Philosophy*, 70, 1973, 5-19), I have used the name DDL^{F} for the system $DDL^{\exists}$ elsewhere.]

To which of the systems belong the following formulas as theorems:

(OR) $(O(A/C) \wedge O(A/D)) \to O(A\,/C\vee D)$

(DR) $O(A/C\vee D) \to (O(A/C) \vee O(A/D))$

Solution:

This is a derivation of (OR) in $DDL^{\forall}$:

(Or)

$$\dfrac{\dfrac{\dfrac{O\ (A/C)}{O\ (A/(C\vee D)\wedge C)}\,(\text{ExtC})}{O\ (C\to A/(C\vee D))}\,(\text{Cond}) \qquad \dfrac{\dfrac{O\ (A/D)}{O\ (A/(C\vee D)\wedge D)}\,(\text{ExtC})}{O\ (D\to A/(C\vee D))}\,(\text{Cond})}{\dfrac{O\ ((C\to A)\wedge(D\to A)/(C\vee D))}{O\ (A/C\vee D)}\,(\text{CExt})}\,(\text{DC})$$

E.g. the following set I proves that (OR) is not valid for definition (b): Let $I=\{(C\to A)\wedge B, (D\to A)\wedge\neg B\}$. Then $I\perp\neg C = I\perp\neg D = I\perp\neg(C\vee D) = \{\{C\to A)\wedge B\},\{D\to A)\wedge\neg B\}\}$, so $O(A/C)$ and $O(A/D)$ are true (for some set it suffices to add C resp. D to make A derivable), but not $O(A\,/C\vee D)$ (for no set it suffices to add just $C\vee D$ to derive A).

This is a derivation of (DR) in $DDL^{\exists}$ (we prove equivalently that $O(A/C\vee D) \to (P(\neg A/C) \to O(A/D))$:

(DR)

$$\dfrac{\dfrac{\dfrac{\dfrac{O\ (A/C\vee D) \qquad \dfrac{\dfrac{P\ (\neg A/C)}{P\ (C\to\neg A/C\vee D)}\,(\text{CCMon})}{P\ ((C\wedge\neg A)\vee D/C\vee D)}\,(\text{CExt})}{O\ (A/(C\wedge\neg A)\vee D)}\,(\text{RMon})}{O\ (A\wedge\neg(C\wedge\neg A)/(C\wedge\neg A)\vee D)}\,(\text{RW})}{O\ (A\wedge D/(C\wedge\neg A)\vee D)}\,(\text{CExt})}{O\ (A/D)}\,(\text{CCMon})$$

Note 1: Notice that (CCMon) $O(A\wedge D/C) \to O(A/C\wedge D)$ is equivalent to (Cond^{P}) $P(A\,/C\wedge D) \to P(D\to A\,/C)$.

Note 2: (RW) is the 'rule of ceteris paribus monotonicity' derivable from M and CExt: if $\vdash_{PL} A\to B$ then $\vdash_{DDL} O(A/C)\to O(B/C)$.

E.g. the following set I proves that (DR) is not valid for def. (a): Let $I=\{(B\to A)\wedge F, B\wedge F\wedge\neg C, (B\to A)\wedge\neg F, B\wedge\neg F\wedge\neg D\}$. Then $I\perp\neg(C\vee D) = \{\{(B\to A)\wedge F, B\wedge F\wedge\neg C\}, \{(B\to A)\wedge\neg F, B\wedge\neg F\wedge\neg D\}\}$ which makes true $O(A/C\vee D)$ since A derives from both sets. But $I\perp\neg C = \{\{(B\to A)\wedge F\}, \{(B\to A)\wedge\neg F, B\wedge\neg F\wedge\neg D\}\}$ and $I\perp\neg D = \{\{(B\to A)\wedge F, B\wedge F\wedge\neg C\}, \{(B\to A)\wedge\neg F\}\}$, in the first case the left set does not derive A, and neither does the right set in the second case, and so $O(A/C)$ and $O(A/D)$ are false.

□

<u>*Extra question:*</u> What would we have to assume of the set of imperatives (imperative contents) in order to make all DSDL3-axioms (including rational monotony RMon) hold for the operator as defined by (a)?

<u>*Solution:*</u>

The construction of a counterexample for (RMon) seems to *rely* (can we prove it?) on having at least two remainder sets, of which one contains at least two formulas individually necessary to derive *A*, and of which one formula gets removed for the logically stronger circumstances.

A radical way to eliminate such counterexamples is therefore to completely remove the possibility of having several remainders, for *arbitrary* circumstances *C*. This is possible! To achieve this, all imperative contents in *I* must be logically chained, i.e. for all $A,B \in I$, we have either $\vdash_{PL} A \rightarrow B$ or $\vdash_{PL} B \rightarrow A$.

Proof: Let $I = X \cup \{A,B\}$. If the remainder set is to be unique for arbitrary circumstances, then it must also be unique for $\neg A \vee \neg B$. Then $X \cup \{A,B\} \perp \neg((\neg A \vee \neg B))$ can at most be $\{X \cup \{A\}\}$ or $\{X \cup \{B\}\}$. Then (in the first case) *B* or (in the second case) *A* must be inconsistent with $(\neg A \vee \neg B)$, for otherwise (in the first case) $\{Y \cup \{B\}\}$ or (in the second case) $\{Y \cup \{A\}\}$ would also be in $X \cup \{A,B\} \perp \neg((\neg A \vee \neg B))$ for some $Y \subseteq X$. So either $\{\neg A \vee \neg B\} \vdash_{PL} \neg B$ or $\{\neg A \vee \neg B\} \vdash_{PL} \neg A$ which means equivalently that either $\{B\} \vdash_{PL} A \wedge B$ or $\{A\} \vdash_{PL} A \wedge B$, which equivalently means that either $\vdash_{PL} B \rightarrow A$ or $\vdash_{PL} A \rightarrow B$.

Observation: For a logically chained set *I* of imperative contents we have that all obligations in some circumstances *C* are determined by the logically strongest $A \in I$ that is consistent with *C*. This is very similar to the 'system of spheres' defined by a preference relation in e.g Hansson's standard dyadic deontic semantics, where $O(A/C)$ holds if *A* is true in all *C*-worlds in the "highest" sphere with a nonempty intersection with *C*. A chained set *I* is perhaps best imagined to consist of contrary-to-duty imperative-contents $\{A_1, \neg A_1 \rightarrow A_2 , (\neg A_1 \wedge \neg A_2) \rightarrow A_3 , \ldots\}$, which then corresponds to such a system of spheres.

Open Question: The assumption that all imperative contents are chained is a rather heavy restriction on 'real-life' imperatives. Is there some less radical way to restrict the imperatives in order to eliminate the counterexamples for (RMon)?

Finally a remark: Multiple remainder sets for some circumstances *C* correspond to a conflict of norms, or a normative dilemma for these circumstances. To avoid conflicts (to make the remainder set unique), a very old idea from legal and moral philosophy is to use a priority ordering of the imperatives (in case of a conflict, less important imperatives are overridden by more important ones). How this works is described in Jörg Hansen: Deontic Logics for Prioritized Imperatives, AI&L 14 (2006), 1-34.