Non-monotonic Reasoning With Various Kinds of Preferences

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Abstract

We are interested in systems which do not prescribe one single kind of preference, but in which varying kinds of preferences can be used simultaneously. In such systems it is essential to know the interaction among the kinds of preferences being used, and we therefore introduce and study a nonmonotonic logic to reason about sixteen strict and non-strict kinds of preferences, including *ceteris paribus* preferences. Moreover, we study "distinguished" preference orders based on specificity principles by showing when these distinguished preference orders are unique, and by presenting algorithms to calculate the distinguished preference orders.

1 Introduction

We are interested in systems which do not prescribe one single kind of preference, but in which varying kinds of preferences can be used simultaneously. For example, when buying a car a user may specify a weak preference for five-doors, a ceteris paribus preference for a European car, and a non-strict preference for blue. Alternatively, a group decision making system may be combining preferences from several agents using distinct kinds of preference statements to communicate their desires. In such systems it will be essential to know the interaction among kinds of preferences.

Preference logic can be used to study the interaction among various kinds of preferences both in monotonic logic for the definition of preference statements as well as in non-monotonic mechanisms for the way preference statements are used when reasoning under uncertainty. Several propositional preference logics have been introduced and studied, such as for example logics for ceteris paribus preferences (comparative statements under the proviso "all else being equal") and weak preferences (e.g., p is preferred to q if the best $p \vee q$ are $\neg q$). For some of the preference logics non-monotonic reasoning mechanisms have been developed, most notably based on specificity principles such as gravitation towards the ideal, also known as System Z, or compactness. To study the interaction among kinds of preferences, we are interested in developing a non-monotonic preference logic for various kinds of preferences.

In recent work [Kaci and van der Torre, 2005], we develop algorithms for a non-monotonic preference logic for four kinds of preferences without a ceteris paribus proviso. In this paper we raise the following four questions:

1. How can we extend the various kinds of preferences with a ceteris paribus proviso?
2. How can we extend the non-monotonic reasoning mechanism based on specificity principles to deal with all kinds of preferences (both strict and non-strict)?
3. Are the distinguished preference orders unique?
4. Which algorithms can be defined to calculate these unique preference orders?

Concerning the ceteris paribus proviso, as far as we know this has only be defined for strong preferences (e.g., p is preferred to q if each p is preferred to all q). In the most general setting we know of, Hansson [1996] defined a ceteris paribus proviso as a filter over the pairs of worlds being compared by a strong preference. This works fine for such strong preferences, but it is less clear how this can be used for weak preferences. We therefore use the more restricted approach of Doyle and Wellman [1994], who define a contextual equivalence relation to interpret the ceteris paribus proviso as "all else being similar". The equivalence relation is again used as a kind of filter (representing similarity).

Concerning the non-monotonic reasoning, we distinguish minimal and maximal specificity principles. Non-monotonic consequences of a logical theory are defined as all formulas which are true in the distinguished models of the theory. An attractive property is case in which there is only one distinguished model, because in that case it can be decided whether a formula non-monotonically follows from a logical theory by calculating the unique distinguished model, and testing whether the formula is satisfied by the distinguished model. Likewise, all non-monotonic consequences can be found by calculating the unique distinguished model and characterizing all formulas satisfied by this model. Therefore we are interested in developing algorithms for unique distinguished models.

The layout of this paper is as follows. In Section 2 we introduce the logic of preferences, and in Section 3 we extend it with specificity principles. Section 4 introduces algorithms to calculate distinguished preference orders.

2 Logic of preferences

To describe ceteris paribus preferences, we use a general construction proposed by Doyle and Wellman [1994]. They base their ceteris paribus preferences on a notion of contextual equivalence.

Definition 1 (Contextual equivalence) *[Doyle and Wellman, 1994, Def.4] Let W be a set of interpretations of a propositional language, and $\xi(W)$ be the set of equivalence relations on W, i.e., the set of reflexive, transitive and symmetric relations $\subseteq W \times W$. A contextual equivalence on W is a function $\eta : 2^{2^W} \rightarrow \xi(W)$ assigning to each set of propositional formulas $\{p, q, \ldots\}$ an equivalence relation $\eta(p, q, \ldots)$. We write $w \equiv w' \mod_\eta p, q, \ldots$.*

If the equivalence relation is the universal relation, i.e., an equivalence relation with only one equivalence class, then the ceteris paribus preference reduces to a strong condition (p is preferred to q when each $p \wedge \neg q$ is preferred to all $\neg p \wedge q$).

Definition 2 (Comparative greatness) *[Doyle and Wellman, 1994, Def.5] Let $\succeq$ be a total pre-order over W, that is, a complete, reflexive and transitive relation over W. We say that p is weakly greater than q iff $w \succeq w'$ whenever $w \models p \wedge \neg q$, $w' \models \neg p \wedge q$, and $w \equiv w' \mod_\eta p \wedge \neg q, \neg p \wedge q$.*

The following proposition shows that this definition reduces a strong preference with a ceteris paribus proviso to a set of strong preferences for each equivalence class of the equivalence relation. The advantage of this reduction is that the ceteris paribus proviso can be used for other kinds of preferences, such as weak preferences.

Proposition 1 *Assume a finite set of propositional atoms, and let $\epsilon(\eta, p, q)$ be the set of propositions which are true in all worlds of an equivalence class of $\eta(p, q)$, but false in all others $\{r \mid \exists w \forall w' (w \equiv w' \mod_\eta p, q \text{ iff } w \models r)\}$. We have that p is weakly greater than q iff for all propositions $c \in \epsilon(\eta, p \wedge \neg q, \neg p \wedge q)$, we have that $w \geq w'$ whenever $w \models p \wedge \neg q \wedge c$, $w' \models \neg p \wedge q \wedge \neg c$.*

The logical language extends propositional logic with sixteen kinds of preferences, based on four binary choices. First, a preference can be strict or non-strict, represented by $>$ and $\geq$, respectively. Moreover, the left hand side and the right hand side of the preference can each be indexed by either a small m for 'pessimistic' $\min$ or a capital M for an 'optimistic' $\max$, as is explained in the semantics in Definition 4. In [Kaci and van der Torre, 2005] we introduce and motivate the following terminology.

$p \ {}^{M}{>}^{M} q$: **Locally optimistic:** optimistic about p and q.

$p \ {}^{m}{>}^{m} q$: **Locally pessimistic:** pessimistic about p and q.

$p \ {}^{M}{>}^{m} q$: **Opportunistic:** optimistic about p, pessimistic about q.

$p \ {}^{m}{>}^{M} q$: **Careful:** pessimistic about p, optimistic about q.

Finally, the preference can be either evaluated without or with a ceteris paribus proviso, where the latter is represented by an index c. For example, $p \ {}^{m}{>}_{c}^{m} q$ is a strict preference of p over q where left and right hand side are evaluated according to $\min$ and with a ceteris paribus proviso.

Definition 3 (Language) *Given a finite set of propositional atoms $A = \{a_1, \ldots, a_n\}$, we define the set L_0 of propositional formulas and the set L of preference formulas as the smallest set satisfying the following.*

$L_0 \ni p, q$**:** $a_i \mid (p \wedge q) \mid \neg p$

$L \ni \phi, \psi$**:** $p \ {}^{x}{>}^{y} q \mid p \ {}^{x}{\geq}^{y} q \mid p \ {}^{x}{>}_{c}^{y} q \mid p \ {}^{x}{\geq}_{c}^{y} \mid \neg \phi \mid (\phi \wedge \psi)$ *for* $x, y \in \{m, M\}$

Disjunction $\vee$, material implication $\supset$ and equivalence $\leftrightarrow$ are defined as usual. Moreover, we define conditionals in terms of preferences by $p \ {}^{m}{\rightarrow}^{m} q =_{def} p \wedge q \ {}^{m}{>}^{m} p \wedge \neg q$, etc. We abbreviate formulas using the following order on logical connectives: $\neg \mid \vee, \wedge \mid > \mid \supset, \leftrightarrow$. For example, $p \vee q > r \supset s$ is interpreted as $((p \vee q) > r) \supset s$.

In our semantics, a ceteris paribus preference is evaluated as a set of preference statements, for each equivalence class of the equivalence relation. Like Doyle and Wellman, we define preferences of p over q as preferences of $p \wedge \neg q$ over $q \wedge \neg p$. This is standard and known as von Wright's expansion principle [von Wright, 1963]. Additional clauses may be added for the cases in which sets of worlds are nonempty, to prevent the satisfiability of preferences like $p > \top$ and $p > \bot$. In this paper we do not consider this borderline condition.

Definition 4 (Monotonic semantics) *Let A be a finite set of propositional atoms, L a propositional logic based on A, W the set of propositional interpretations of L, $\succeq$ a total pre-order on W, and η a contextual equivalence relation on W. We write $w \succ w'$ for $w \succeq w'$ without $w' \succeq w$. Moreover, if $x = M$ then we write $x(p, \succeq) = \max(p, \succeq)$ for $\{w \in W \mid w \models p, \forall w' \in W : w' \models p \Rightarrow w \succeq w'\}$, and analogously when $x = m$ we write $x(p, \succeq) = \min(p, \succeq)$ for $\{w \in W \mid w \models p, \forall w' \in W : w' \models p \Rightarrow w' \succeq w\}$.*

$\succeq, \eta \models p \ {}^{x}{>}^{y} q$ if and only if $\forall w \in x(p \wedge \neg q, \succeq)$ and $\forall w' \in y(\neg p \wedge q, \succeq)$ we have $w \succ w'$

$\succeq, \eta \models p \ {}^{x}{\geq}^{y} q$ if and only if $\forall w \in x(p \wedge \neg q, \succeq)$ and $\forall w' \in y(\neg p \wedge q, \succeq)$ we have $w \succeq w'$

$\succeq, \eta \models p \ {}^{x}{>}_{c}^{y} q$ if and only if $\forall c \in \epsilon(\eta, p \wedge \neg q, \neg p \wedge q)$, $\forall w \in x(p \wedge \neg q \wedge c, \succeq)$ and $\forall w' \in y(\neg p \wedge q \wedge c, \succeq)$ we have $w \succ w'$

$\succeq, \eta \models p \ {}^{x}{\geq}_{c}^{y} q$ if and only if $\forall c \in \epsilon(\eta, p \wedge \neg q, \neg p \wedge q)$, $\forall w \in x(p \wedge \neg q \wedge c, \succeq)$ and $\forall w' \in y(\neg p \wedge q \wedge c, \succeq)$ we have $w \succeq w'$

We sometimes assume a fixed η, when we write $\succeq \models_\eta \phi$ for $\succeq, \eta \models \phi$. Moreover, logical notions are defined as usual, in particular:

- *$\eta \models \phi$ iff for all $\succeq$, we have $\succeq, \eta \models \phi$,*
- *$S \models_\eta \phi$ iff for all $\succeq$ s.t. $\succeq, \eta \models S$, we have $\succeq, \eta \models \phi$.*

Example 1 illustrates the logic of preferences.

Example 1 *$\models p \ {}^{M}{>}^{M} q \leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q) \ {}^{M}{\rightarrow}^{M} p$, which expresses a well-known relation between a defeasible conditional ${}^{M}{\rightarrow}^{M}$ and preferences ${}^{M}{>}^{M}$. Moreover, we have $\models p \ {}^{m}{>}^{M} q \supset p \ {}^{M}{>}^{M} q$, which expresses that strong preferences ${}^{m}{>}^{M}$ imply weak preferences ${}^{M}{>}^{M}$. Finally, we have $\models p \ {}^{m}{>}^{M} q \supset p \ {}^{m}{>}_{c}^{M} q$, which expresses that preferences without ceteris paribus proviso imply ceteris paribus preferences.*

$\mathcal{P}_{mm}$		$\mathcal{P}_{mM}$		$\mathcal{P}_{Mm}$		$\mathcal{P}_{MM}$	
least no	most – yes [Dubois *et al.*, 2004b]	least – yes [Benferhat and Kaci, 2001]	most yes	least no	most no	least – yes [Pearl, 1990] [Benferhat *et al.*, 1992]	most no

Table 1: Uniqueness of distinguished pre-orders

3 Non-monotonic logic of preferences

A preference specification consists of sets of preferences.

Definition 5 (Preference specification) *Let $\mathcal{P}_{\rhd}$ be a set of preferences of the form $\{p_i \rhd q_i : i = 1, \cdots, n\}$. A preference specification is a tuple $\langle \mathcal{P}_{\rhd} \mid \rhd \in \{ {}^{x}{>}^{y}, {}^{x}{\geq}^{y}, {}^{x}{>}^{y}_{c}, {}^{x}{\geq}^{y}_{c} \mid x, y \in \{m, M\}\}\rangle$. A total pre-order $\succeq$ together with a contextual equivalence function η is a model of $\mathcal{P}_{\rhd}$ iff $\succeq, \eta$ satisfies each preference $p_i \rhd q_i$ in $\mathcal{P}_{\rhd}$.*

The following definition illustrates how a preference order can also be represented by a well ordered partition of W. This is an equivalent representation, in the sense that each preference order corresponds to one ordered partition and vice versa. This equivalent representation as an ordered partition makes the definition of the non-monotonic semantics easier to read.

Definition 6 (Ordered partition) *A sequence of sets of worlds of the form $(E_1, \ldots, E_n)$ is an ordered partition of W iff $\forall i$, E_i is nonempty, $E_1 \cup \cdots \cup E_n = W$ and $\forall i, j$, $E_i \cap E_j = \emptyset$ for $i \neq j$. An ordered partition of W is associated with pre-order $\succeq$ on W iff $\forall \omega, \omega' \in W$ with $\omega \in E_i, \omega' \in E_j$ we have $i \leq j$ iff $\omega \succeq \omega'$.*

Shoham [1987] characterizes non-monotonic reasoning as a mechanism that selects a subset of the models of a set of formulas, which we call distinguished models in this paper. Shoham calls these models "preferred models", but we do not use this terminology as this meta-logical terminology may be confused with preferences in the logical language and preference orders in the semantics.

In this paper we compare total pre-orders based on the so-called specificity principle. The minimal specificity principle is gravitating towards the least specific pre-order, while the maximal specificity principle is gravitating towards the most specific pre-order. These have been used in non-monotonic logic to define the distinguished model of a set of conditionals of the kind ${}^{M}{\rightarrow}^{M}$, sometimes called defeasible conditionals.

Definition 7 (Minimal/Maximal specificity principle) *Let $\succeq$ and $\succeq'$ be two total pre-orders on a set of worlds W represented by ordered partitions $(E_1, \cdots, E_n)$ and $(E'_1, \cdots, E'_n)$ respectively. We say that $\succeq$ is at least as specific as $\succeq'$, written as $\succeq \sqsubseteq \succeq'$, iff $\forall \omega \in W$, if $\omega \in E_i$ and $\omega \in E'_j$ then $i \leq j$. $\succeq$ is said to be the least (resp. most) specific pre-order among a set of pre-orders $\mathcal{O}$ if there is no $\succeq'$ in $\mathcal{O}$ s.t. $\succeq' \sqsubset \succeq$, i.e., $\succeq' \sqsubseteq \succeq$ without $\succeq \sqsubseteq \succeq'$ (resp. $\succeq \sqsubset \succeq'$).*

The following example illustrates minimal and maximal specificity principles.

Example 2 *Consider the rule $p \ {}^{x}{\rightarrow}^{y} q$. Applying the minimal specificity principle on $p \ {}^{M}{\rightarrow}^{M} q$ or $p \ {}^{m}{\rightarrow}^{M} q$ gives the following model $\succeq = (\{pq, \neg pq, \neg p\neg q\}, \{p\neg q\})$. The preferred worlds in this model are those which do not violate the rule. More precisely pq belongs to the set of preferred worlds since it satisfies the rule but $\neg pq$ and $\neg p\neg q$ are preferred too since they do not violate the rule even if they do not satisfy it. Now applying the maximal specificity principle on $p \ {}^{m}{\rightarrow}^{m} q$ gives the following model $\succeq' = (\{pq\}, \{\neg pq, p\neg q, \neg p\neg q\})$. We can see that the preferred worlds are those which only satisfy the rule.*

Unique distinguished models have been computed for preferences of type ${}^{M}{>}^{M}$, ${}^{m}{>}^{M}$ and ${}^{m}{>}^{m}$ considered individually. Table 1 summarizes these results. Hence, no unique distinguished models can be calculated for careful preferences, which does not seem very problematic since this kind of preferences is rarely used, as they seem too weak. However it may be useful when all other preference types give an empty set of models [van der Torre and Weydert, 2001]:

Example 3 *Let j and f be two propositional variables which stand for marriage with John and Fred, respectively. Let $\mathcal{P}_{{}^{x}{>}^{y}} = \{j \ {}^{x}{>}^{y} \neg j, f \ {}^{x}{>}^{y} \neg f, \neg(j \wedge f) \ {}^{x}{>}^{y} (j \wedge f)\}$ be a set of Sue's preferences about its marriage with John or Fred. The first constraint means that Sue prefers to be married to John over not being married to him. The second constraint means that Sue prefers to be married to Fred over not being married to him and the last constraint means that Sue prefers not to be married to both. There is no pre-order satisfying any of the sets $\mathcal{P}_{{}^{M}{>}^{M}}$, $\mathcal{P}_{{}^{m}{>}^{M}}$ and $\mathcal{P}_{{}^{m}{>}^{m}}$ while the following pre-order $(\{j\neg f, \neg jf\}, \{jf, \neg j\neg f\})$ satisfies $\mathcal{P}_{{}^{M}{>}^{m}}$.*

Since the uniqueness of the distinguished model is not satisfied for ${}^{M}{>}^{m}$ preferences, in the remainder of this paper we only focus on locally optimistic, locally pessimistic and careful preferences. We consider in [Kaci and van der Torre, 2005] preferences of different types given together and shown that there is a unique least specific model for ${}^{m}{>}^{M}$ and ${}^{M}{>}^{M}$, and a unique most specific model for ${}^{m}{>}^{M}$ and ${}^{m}{>}^{m}$. As far as we know, the following questions have not been addressed yet.

1. Is a least specific pre-order of preference specification $\langle \mathcal{P}_{\rhd} | \rhd \in \{ {}^{x}{>}^{y}, {}^{x}{\geq}^{y}, {}^{x}{>}^{y}_{c}, {}^{x}{\geq}^{y}_{c} | x \in \{m, M\}, y = M\}\rangle$ unique?
2. Is a most specific model of preference specification $\langle \mathcal{P}_{\rhd} | \rhd \in \{ {}^{x}{>}^{y}, {}^{x}{\geq}^{y}, {}^{x}{>}^{y}_{c}, {}^{x}{\geq}^{y}_{c} | x = m, y \in \{m, M\}\}\rangle$ unique?
3. How can we define a model of preference specification $\langle \mathcal{P}_{\rhd} \mid \rhd \in \{ {}^{x}{>}^{y}, {}^{x}{\geq}^{y}, {}^{x}{>}^{y}_{c}, {}^{x}{\geq}^{y}_{c} \mid x, y \in \{m, M\}\}\rangle$ based on the distinguished models computed in items 1 and 2.

In the following section the first two questions are answered positively by providing algorithms for these unique most specific and most general pre-orders. Moreover, the third question is answered by presenting a way to combine these two unique pre-orders.

4 Algorithms for mixed preferences

In this section we provide algorithms to calculate minimal and maximal specific pre-orders. We first replace ceteris paribus preferences by sets of ordinary preferences without a ceteris paribus proviso, by generalizing proposition 1 for the other kinds of preferences.

Proposition 2 *Let $\epsilon(\eta, p, q)$ be the set of propositions which are true in all worlds of an equivalence class of $\eta(p,q)$, but false in all others $\{r \mid \exists w \forall w' (w \equiv w' \mod_\eta p, q$ iff $w \models r)\}$, $\rhd \in \{ {}^x\!\!>^y, {}^x\!\!\geq^y \mid x, y \in \{m, M\}\}$, and $\rhd_c \in \{ {}^x\!\!>^y_c, {}^x\!\!\geq^y_c \mid x, y \in \{m, M\}\}$ for the same connective extended with a ceteris paribus proviso. We have $\succeq, \eta \models p \rhd_c q$ iff for all propositions $c \in \epsilon(\eta, p \wedge \neg q, \neg p \wedge q)$, we have $\succeq, \eta \models p \wedge \neg q \wedge c \rhd \neg p \wedge q \wedge c$.*

Consequently, we can restrict ourselves to the eight types of preferences without ceteris paribus clauses. In these calculations, we do not refer to the contextual equivalence relation η anymore.

4.1 A distinguished model for locally optimistic and careful preferences

We consider the following preference specifications $\langle \mathcal{P}_\rhd \mid \rhd \in \{ {}^x\!\!>^y, {}^x\!\!\geq^y, {}^x\!\!>^y_c, {}^x\!\!\geq^y_c \mid x \in \{m, M\}, y = M\}\rangle$, and since we assume that preference specifications with ceteris paribus provisos have been reduced to sets of preferences without such provisos, we consider the following four sets of preferences:
$\mathcal{P}_{M_> M} = \{C_{i_1} : p_{i_1} \ {}^M\!\!>^M q_{i_1}\}, \mathcal{P}_{M_\geq M} = \{C_{i_2} : p_{i_2} \ {}^M\!\!\geq^M q_{i_2}\}$,
$\mathcal{P}_{m_> M} = \{C_{i_3} : p_{i_3} \ {}^m\!\!>^M q_{i_3}\}, \mathcal{P}_{m_\geq M} = \{C_{i_4} : p_{i_4} \ {}^m\!\!\geq^M q_{i_4}\}$.
Moreover, we refer to the constraints of these preferences by $\overline{\mathcal{C}} = \bigcup_{k=1,\cdots,4} \{\overline{\mathcal{C}}_{i_k} = (L(C_{i_k}), R(C_{i_k}))\}$, where the left and right hand side of these constraints are $L(C_{i_k}) = |p_{i_k} \wedge \neg q_{i_k}|$ and $R(C_{i_k}) = |\neg p_{i_k} \wedge q_{i_k}|$, respectively. Algorithm 1 computes the least specific model of locally optimistic and careful preferences with and without ceteris paribus, strict and non-strict. The basic idea of the algorithm is to construct the least specific pre-order by calculating the sets of worlds of the ordered partition, going from the ideal to the worst worlds.

Let us first explain the algorithm, then we illustrate it by an example and finally we show that the distinguished model computed by the algorithm is the unique least specific one. At each step of the algorithm, we look for worlds which can have the actual highest ranking in the preference order. This corresponds to the actual minimal value l. These worlds are those which do not falsify any constraint in $\overline{\mathcal{C}}$. We first put in E_l worlds which do not falsify any strict preferences. These worlds are those which do not appear in the right hand sides of the strict preferences $\overline{\mathcal{C}}_{i_1}$ and $\overline{\mathcal{C}}_{i_3}$. Now we remove from E_l worlds which falsify constraints of the non-strict preferences $\overline{\mathcal{C}}_{i_2}$ and $\overline{\mathcal{C}}_{i_4}$. Locally optimistic constraints $\overline{\mathcal{C}}_{i_2}$ are violated if $L(C_{i_2}) \cap E_l = \emptyset$ and $R(C_{i_2}) \cap E_l \neq \emptyset$, while the careful constraints $\overline{\mathcal{C}}_{i_4}$ are violated if $L(C_{i_4}) \not\subseteq E_l$ and $R(C_{i_4}) \cap E_l \neq \emptyset$. Once E_l is fixed, satisfied constraints are removed. Note that constraints $\overline{\mathcal{C}}_{i_k}$ s.t. $k \in \{1, 2\}$ are satisfied if $L(C_{i_k}) \cap E_l \neq \emptyset$ since in this case, worlds of $R(C_{i_1})$ are necessarily in E_h with $h > l$ and worlds of

Algorithm 1: Handling locally optimistic and careful preferences.

begin
$l \leftarrow 0$;
while $W \neq \emptyset$ **do**
– $l \leftarrow l + 1, j \leftarrow 1$;
/** strict constraints **/
– $E_l = \{\omega : \forall \overline{\mathcal{C}}_{i_1}, \overline{\mathcal{C}}_{i_3} \in \overline{\mathcal{C}}, \omega \notin R(C_{i_1}) \cup R(C_{i_3})\}$;
while $j = 1$ **do**
$j \leftarrow 0$;
for *each* $\overline{\mathcal{C}}_{i_2}$ *and* $\overline{\mathcal{C}}_{i_4}$ *in* $\overline{\mathcal{C}}$ **do**
/** non-strict constraints **/
if *$(L(C_{i_2}) \cap E_l = \emptyset$ and $R(C_{i_2}) \cap E_l \neq \emptyset)$ or $(L(C_{i_4}) \not\subseteq E_l$ and $R(C_{i_4}) \cap E_l \neq \emptyset)$* **then**
$E_l = E_l - R(C_{i_k})$;
$j \leftarrow 1$
if $E_l = \emptyset$ **then** Stop (inconsistent constraints);
– remove from W elements of E_l ;
/** remove MM satisfied constraints **/
– remove from $\overline{\mathcal{C}}$ constraints $\overline{\mathcal{C}}_{i_k}$ $k \in \{1, 2\}$ such that $L(C_{i_k}) \cap E_l \neq \emptyset$;
/** update mM constraints **/
– replace constraints $\overline{\mathcal{C}}_{i_k}$ ($k \in \{3, 4\}$) by $(L(C_{i_k}) - E_l, R(C_{i_k}))$;
/** remove satisfied mM constraints **/
– remove from $\overline{\mathcal{C}}$ constraints $\overline{\mathcal{C}}_{i_k}$ ($k \in \{3, 4\}$) with empty $L(C_{i_k})$.
return $(E_1, \cdots, E_l)$
end

$R(C_{i_2})$ are in $E_{h'}$ with $h' \geq l$. However careful constraints $\overline{\mathcal{C}}_{i_k}$ (for $k \in \{3, 4\}$) are satisfied only when $L(C_{i_k}) \subseteq E_l$ otherwise they should be replaced by $(L(C_{i_k}) - E_l, R(C_{i_k}))$.

The least specific criterion can be checked by construction. At each step l we put in E_l all worlds which do not appear in any $R(C_{i_k})$ for $k \in \{1, 3\}$ and which are not yet put in some $E_{l'}$ with $l' < l$. If $\omega \in E_l$ then it necessarily falsifies some constraints which are not falsified by worlds of $E_{l'}$ for $l' < l$. If we would put some ω of E_l in $E_{l'}$ with $l' < l$ then we get a contradiction. To show the uniqueness of the least specific model of $\mathcal{P}$, we follow the line of the proof given in [Benferhat *et al.*, 1999].

Lemma 1 *The total pre-order computed by Algorithm 1 belongs to the set of least specific models of $\mathcal{P}$.*

We now define the maximum of two preference orders.

Definition 8 *Let $\succeq$ and $\succeq'$ be two preference orders represented by their well ordered partitions $(E_1, \cdots, E_n)$ and $(E'_1, \cdots, E'_{n'})$ respectively. We define the $\mathcal{MAX}$ operator by $\mathcal{MAX}(\succeq, \succeq') = (E''_1, \cdots, E''_{min(n,n')})$, such that $E''_1 = E_1 \cup E'_1$ and $E''_k = (E_k \cup E'_k) - (\bigcup_{i=1,\cdots,k-1} E''_i)$ for $k = 2, \cdots, min(n, n')$, and the empty sets E''_k are eliminated by renumbering the non-empty ones in sequence.*

Lemma 2 proves the uniqueness of the least specific model of $\mathcal{P}$.

Lemma 2 *If there is a minimal specific model of $\mathcal{P}$, then it is unique.*

Proof. *Let $\mathcal{M}(\mathcal{P})$ be the set of models of $\mathcal{P}$. We first show that $\mathcal{MAX}(\succeq, \succeq') \in \mathcal{M}(\mathcal{P})$ (1). Let $\succeq = (E_1, \cdots, E_h)$, $\succeq' = (E'_1, \cdots, E'_{h'})$, $\succeq'' = (E''_1, \cdots, E''_{min(h,h')})$, and $p \ {}^{M}\!\!\geq^{M} q \in \mathcal{P}$. $\succeq, \succeq' \in \mathcal{M}(\mathcal{P})$, i.e., $\succeq \models p \ {}^{M}\!\!\geq^{M} q$ and $\succeq' \models p \ {}^{M}\!\!\geq^{M} q$. In other words, $\max(p \wedge \neg q, \succeq) \subseteq E_i$ and $\max(\neg p \wedge q, \succeq) \subseteq E_j$ s.t. $i \leq j$ and $\max(p \wedge \neg q, \succeq') \subseteq E'_k$ and $\max(\neg p \wedge q, \succeq') \subseteq E'_l$ s.t. $k \leq l$. Following Definition 8, $\max(p \wedge \neg q, \succeq'') \subseteq E''_{\min(i,k)}$ and $\max(\neg p \wedge q, \succeq'') \subseteq E''_{\min(j,l)}$. Since $i \leq j$ and $k \leq l$ we have $\min(i,k) \leq \min(j,l)$. We conclude $\succeq'' \models p \ {}^{M}\!\!\geq^{M} q$. The proofs for the other constraints are analogous and can be found in [Benferhat* et al.*, 1999]. Consequently, $\mathcal{MAX}(\succeq, \succeq') \in \mathcal{M}(\mathcal{C})$.*

Moreover, we have that $\mathcal{MAX}(\succeq, \succeq')$ is less specific than or identical to both $\succeq$ and $\succeq'$ (2), the proof can be found also in [Benferhat et al.*, 1999].*

Finally, we prove that the lemma follows from the two items by contradiction. So suppose that there are two distinct minimal specific orders $\succeq$ and $\succeq'$. Then according to item (1), $\mathcal{MAX}(\succeq, \succeq')$ is also a model of the preference specification and according to item (2), it is less specific than either $\succeq$ or $\succeq'$. Contradiction.

We can now conclude:

Theorem 1 *Algorithm 1 computes the least specific model of $\mathcal{P}$.*

Proof. Following Lemma 1 it computes a preference order which belongs to the set of the least specific models and following Lemma 2, this preference order is unique.

Example 4 *Let p, q and r be three propositional variables which stand respectively for sun, beach and cheap. Suppose that an agent is looking for holidays destination. Let $\mathcal{P} = \langle \mathcal{P}_{\,{}^{M}\!>^{M}}, \mathcal{P}_{\,{}^{m}\!>^{M}}, \mathcal{P}_{\,{}^{m}\!\geq^{M}} \rangle$ be the set of its preferences, where $\mathcal{P}_{\,{}^{M}\!>^{M}} = \{p \wedge q \ {}^{M}\!\!>^{M} \neg(p \wedge q)\}$,*
$\mathcal{P}_{\,{}^{m}\!>^{M}} = \{p \wedge r \ {}^{m}\!\!>^{M} p \wedge \neg r\}$ and
$\mathcal{P}_{\,{}^{m}\!\geq^{M}} = \{p \wedge r \ {}^{m}\!\!\geq^{M} \neg p \wedge r, p \wedge \neg r \ {}^{m}\!\!\geq^{M} \neg p \wedge \neg r\}$.
Applying Algorithm 1[1] gives $E_1 = \{pqr\}$,
$E_2 = \{\neg p \neg qr, \neg pqr, p \neg qr\}$ and
$E_3 = \{\neg p \neg q \neg r, \neg pq \neg r, pq \neg r, p \neg q \neg r\}$.
The computed model means that the preferred destinations are cheap, sunny and where there is a beach, the next preferred ones are cheap but either not sunny or there is no beach. Lastly the least preferred destinations are those which are not cheap whatever there is beach or the place is sunny.

The above algorithm is general. It captures all existing algorithms proposed for handling ${}^{M}\!\!>^{M}$ and ${}^{m}\!\!>^{M}$ separately [Pearl, 1990; Benferhat and Kaci, 2001], mixed preferences ${}^{M}\!\!>^{M}$ and ${}^{m}\!\!>^{M}$ [Kaci and van der Torre, 2005]. This algorithm also generalizes the algorithm of [Benferhat *et al.*, 2001] which captures "equal preferences", denoted $p \ {}^{M}\!\!=^{M} q$, which stands for "all best p worlds are q worlds and all q worlds are p worlds". These equivalences can be represented in our framework by two non-strict preferences $p \ {}^{M}\!\!\geq^{M} q$ and $q \ {}^{M}\!\!\geq^{M} p$, but our non-strict preferences cannot be represented in their framework.

[1]Due to the lack of space, we omit the details of the construction of $\succeq$.

4.2 Locally pessimistic and careful preferences

Algorithm 2 is structurally similar to Algorithm 1.1., and the proof that this algorithm produces the most specific model of these preferences is analogous to the proof of Theorem 2.
Let $\mathcal{P}' = \langle \mathcal{P}_{\rhd} \mid \rhd \in \{ {}^{x}\!\!>^{y}, {}^{x}\!\!\geq^{y}, {}^{x}\!\!>^{y}_{c}, {}^{x}\!\!\geq^{y}_{c} \mid x = m \text{ and } y \in \{m, M\}\}\rangle$. Again we assume that preference specifications with ceteris paribus provisos have been reduced to sets of preferences without such provisos, so we consider the following four sets of preferences:
$\mathcal{P}_{\,{}^{m}\!>^{m}} = \{C_{i_1} : p_{i_1} \ {}^{m}\!\!>^{m} q_{i_1}\}$, $\mathcal{P}_{\,{}^{m}\!\geq^{m}} = \{C_{i_2} : p_{i_2} \ {}^{m}\!\!\geq^{m} q_{i_2}\}$,
$\mathcal{P}_{\,{}^{m}\!>^{M}} = \{C_{i_3} : p_{i_3} \ {}^{m}\!\!>^{M} q_{i_3}\}$, $\mathcal{P}_{\,{}^{m}\!\geq^{M}} = \{C_{i_4} : p_{i_4} \ {}^{m}\!\!\geq^{M} q_{i_4}\}$.
Let $\overline{\mathcal{C}} = \bigcup_{k=1,\cdots,4}\{\overline{\mathcal{C}}_{i_k} = (L(C_{i_k}), R(C_{i_k}))\}$, where $L(C_{i_k}) = |p_{i_k} \wedge \neg q_{i_k}|$ and $R(C_{i_k}) = |\neg p_{i_k} \wedge q_{i_k}|$.

Algorithm 2: Handling locally pessimistic and careful preferences.

begin
  $l \leftarrow 0$;
  while $W \neq \emptyset$ **do**
    $l \leftarrow l + 1, j \leftarrow 1$;
    $E_l = \{\omega : \forall \overline{\mathcal{C}}_{i_1}, \overline{\mathcal{C}}_{i_3}$ in $\overline{\mathcal{C}}, \omega \notin L(C_{i_1}) \cup L(C_{i_3})\}$;
    while $j = 1$ **do**
      $j \leftarrow 0$;
      for *each* $\overline{\mathcal{C}}_{i_2}$ *and* $\overline{\mathcal{C}}_{i_4}$ *in* $\overline{\mathcal{C}}$ **do**
        /** non-strict constraints **/
        if *($L(C_{i_2}) \cap E_l \neq \emptyset$ and $R(C_{i_2}) \cap E_l = \emptyset$) or ($L(C_{i_4}) \cap E_l \neq \emptyset$ and $R(C_{i_4}) \not\subseteq E_l$)* **then**
          $E_l = E_l - L(C_{i_k}), j \leftarrow 1$

  if $E_l = \emptyset$ **then** Stop (inconsistent constraints);
  – Remove from W elements of E_l;
  /** remove mm satisfied constraints **/
  – Remove from $\overline{\mathcal{C}}$ constraints $\overline{\mathcal{C}}_{i_k}$ (for $k \in \{1, 2\}$) s.t. $E_l \cap R(C_{i_k}) \neq \emptyset$;
  /** update mM constraints **/
  – Replace $\overline{\mathcal{C}}_{i_k}$ (for $k \in \{3, 4\}$) in $\overline{\mathcal{C}}$ by $(L(C_{i_k}), R(C_{i_k}) - E_l)$;
  /** remove mM satisfied constraints **/
  – Remove from $\overline{\mathcal{C}}$ constraints $\overline{\mathcal{C}}_{i_k}$ ($k \in \{3, 4\}$) with empty $R(C_{i_k})$;
  return $(E'_1, \cdots, E'_l)$ s.t. $\forall 1 \leq h \leq l, E'_h = E_{l-h+1}$
end

Example 5 *Let $\mathcal{P}' = \langle \mathcal{P}_{\,{}^{m}\!>^{m}}, \mathcal{P}_{\,{}^{m}\!>^{M}}, \mathcal{P}_{\,{}^{m}\!\geq^{M}} \}$ with $\mathcal{P}_{\,{}^{m}\!>^{M}}, \mathcal{P}_{\,{}^{m}\!\geq^{M}}$ as in Example 4 and $\mathcal{P}_{\,{}^{m}\!>^{m}} = \{\neg q \ {}^{m}\!\!>^{m} q\}$.*
Applying Algorithm 2 gives $E'_1 = \{p \neg qr, pqr\}$, $E'_2 = \{\neg p \neg q \neg r, \neg p \neg qr, p \neg q \neg r, pq \neg r\}$ and $E'_3 = \{\neg pq \neg r, \neg pqr\}$. This model means that the preferred destinations are sunny and cheap and the least preferred ones are not sunny and there is a beach whatever they are cheap or not.

Theorem 2 *Let $\mathcal{P} = \langle \mathcal{P}_{\rhd} \mid \rhd \in \{ {}^{x}\!\!>^{y}, {}^{x}\!\!\geq^{y}, {}^{x}\!\!>^{y}_{c}, {}^{x}\!\!\geq^{y}_{c} \mid x = m$ and $y \in \{m, M\}\}\rangle$. Then Algorithm 2 computes the most specific model of $\mathcal{P}$ which is unique.*

Proof (sketch). Follows the same lines as the proof of Theorem 2. It can also be derived from Theorem 2 using symmetry of the two algorithms.

4.3 Careful, locally optimistic and locally pessimistic preferences

To find a distinguished model of the sixteen kinds of preferences given together, we combine the two algorithms. It has been argued in [Benferhat *et al.*, 2002; Dubois *et al.*, 2004a] that, in the context of preference modeling, the minimal specificity principle models constraints which should not be violated while the maximal specificity principle models what is really desired by the agent. In our setting, this combination of the least specific and the most specific models leads to a refinement of the former by the latter.

Definition 9 *Let $\succeq''$ be the result of combining $\succeq$ and $\succeq'$ corresponding to the least specific and the most specific models respectively. Then,*

- *if $\omega \succ \omega$ then $\omega \succ'' \omega'$,*
- *if $\omega \simeq \omega'$ then ($\omega \succeq'' \omega'$ iff $\omega \succeq' \omega'$).*

Example 6 (Continued from Examples 4 and 5) *We have $\succeq''= (E''_1, \cdots, E''_6)$ where $E''_1 = \{pqr\}$, $E''_2 = \{p\neg qr\}$, $E''_3 = \{\neg p\neg qr\}$, $E''_4 = \{\neg pqr\}$, $E''_5 = \{\neg p\neg q\neg r, pq\neg r, p\neg q\neg r\}$ and $E''_6 = \{\neg pq\neg r\}$.*
The best destinations are cheap, sunny and there is a beach there and the least preferred destinations are those where there is a beach however they are neither sunny nor cheap.

5 Concluding remarks

We develop a nonmonotonic logic to reason about the interaction among kinds of preferences in systems in which varying kinds of preferences can be used simultaneously. How to use such a logic is still an open question. In [Kaci and van der Torre, 2005] we propose the use of the preference types locally optimistic, locally pessimistic, opportunistic and careful to guide the choice among these kinds of preferences. In the logic proposed in this paper, an important question is when to choose preferences with or without a ceteris paribus proviso. Another question is how to define the contextual equivalence relation, which can for example be inspired by CP nets or by a more complicated mechanism.

We introduce a logic for sixteen kinds of preferences, where the ${}^{m}{>}^{M}$'s preference is the strongest one while ${}^{M}{\geq}_{c}^{m}$'s preference is the weakest one. Some of the sixteen preference statements – or simple variants of it – have been discussed in the literature. For example, Doyle and Wellman's $p\ {}^{m}{>}_{c}^{M} q$ is the comparative statement usually studied in the logic of preference, see, e.g., [von Wright, 1963; Hansson, 1996]. Sometimes it has been suggested that in formalizing preferences, there is a choice between these two options of ceteris paribus ${}^{m}{>}_{c}^{M}$ and an optimizing preference $p\ {}^{M}{\geq}^{M} q$. However, in our more general framework we define ceteris paribus and optimization as two dimensions of preferences, which can also be combined by $p\ {}^{M}{\geq}_{c}^{M} q$.

We show how specificity principles can be used to determine distinguished pre-orders from a preference specification with either locally optimistic and careful strict and non-strict preferences, or with locally pessimistic and careful strict and non-strict preferences, in both case with and without ceteris paribus proviso. However, in case of opportunistic preferences, such minimal and most specific pre-orders are not unique. Moreover, when locally optimistic and locally pessimistic preferences, the distinguished pre-orders have to be merged again. This suggests that other mechanisms may be developed for nonmonotonic reasoning about preferences.

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