Double preference relations for generalised belief change ⋆

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Abstract

Many belief change formalisms employ plausibility orderings over the set of possible worlds to determine how the beliefs of an agent ought to be modified after the receipt of a new epistemic input. While most such possible world semantics rely on a single ordering, we investigate the use of an additional preference ordering—representing, for instance, the epistemic context the agent finds itself in—to guide the process of belief change. We show that the resultant formalism provides a unifying semantics for a wide variety of belief change operators. By varying the conditions placed on the second ordering, different families of known belief change operators can be captured, including AGM belief contraction and revision, Rott and Pagnucco's severe withdrawal, the systematic withdrawal of Meyer et al, as well as the linear liberation and σ-liberation operators of Booth et al. Our approach also identifies novel classes of belief change operators worthy of further investigation.

Key words: Belief revision, belief removal, belief liberation, severe withdrawal.

⋆ This is an expanded version of a paper initially appearing in the Proceedings of the Sixteenth European Conference on Artificial Intelligence (ECAI 2004) [1].

1 INTRODUCTION

Current formalisms in belief change [2,3] typically employ either a plausibility ordering [4,5] over the set of possible worlds, or an epistemic entrenchment ordering [2] over the set of sentences in an agent's belief set. Operators for change are then defined by manipulation of these orderings after receipt of a new epistemic input. There are many advantages to these approaches – foremost amongst them the guarantee that change will be effected in a principled manner, the provision of an intuitively plausible construction, and a formalism flexible enough to accommodate alternative change strategies and iteration. However, some nuances of belief change are not captured in such an approach. For instance, agents do not usually employ one fixed ordering throughout. Often, different orderings might be used in different contexts such as those requiring greater caution or skepticism; or different orderings might be used based on the assessed reliability of the source of the epistemic inputs. Such a critique is implicit in the work of Cantwell [6] where the notion of *eligibility* adds an extra dimension to belief change. A technical framework that provides tools for belief change operations based on multiple orderings was proposed by Andreka et al. [7] where combination operations for a class of preference relations $\mathcal{P}$ are studied in terms of an additional guiding preference relation. In our current approach, the formalism for belief change (in particular belief removal) we present can be considered to be a special case of the work of Andreka et al. with $\leq$ (over the set of interpretations) being the single preference relation in $\mathcal{P}$, and $\preceq$ (our additional dimension) being the guiding relation.

An intuitive way to understand the second ordering on the set of possible worlds is to think of it as a more stringent assessment of the plausibility of states of affairs. Most rational agents are aware of certain contexts within which their reasoning plays out – certain contexts call for a different assessment of plausibility. For example, while I am moderately sceptical in vetting news reports of the generic kind, I adopt a more critical stance when vetting news reports of the more serious kind, say concerning the impending declaration of a war. Such a treatment is reminiscent of contextualist assessments of epistemic statements [8] where it is understood that agents make knowledge claims relative to some implicit standard for assessing that claim and that different standards will induce differing assessments of the truth of epistemic claims. The contribution of this paper is the unification, in a single formal framework, of a large class of belief change operators by a method that employs two preference orderings over the set of possible worlds. It enables us to view belief change as the manipulation, by the agent, of assessments of plausibility of epistemic states of affairs in different contexts.

The plan of the paper is as follows. After laying down some technical preliminaries, in Section 2 we establish the foundations of our framework for removal with a semantic definition and an axiomatic characterisation. The formal definition of removal provided here allows us to show how this framework can be used to peform belief change in different contexts. In Section 3 we study the class of belief removal operators obtained when the second ordering $\preceq$ is transitive. Section 4 builds up to a characterisation of *AGM contraction* [9] via sub-classes of belief removal operators satisfying the standard properties known as Vacuity, Inclusion and Recovery. Section 5 shows that important classes of *belief liberation* operators [10] can be captured in our framework. Section 6 isolates various classes of removal operators related to, and including, *systematic withdrawal* [11]. Section 7 shows that the limiting cases correspond to *AGM revision* [9] and *severe withdrawal* [12], while Section 9 concludes with some pointers to future work. Finally, the formal proofs of all results are collated in an appendix at the end of the paper.

We assume a finitely generated propositional language L equipped with the usual constants, boolean operators and a classical Tarskian consequence relation Cn. Although the finiteness assumption is a limitation, it is often made in the context of logic-based Artificial Intelligence (cf. [5,13]) when (i) it does not detract from the basic principles being investigated, and (ii) it ensures that the proofs are simplified considerably. Both conditions are applicable here. The interested reader is also referred to the work of Gabbay and Schlechta [14] where the initial version of the work presented here [1] is extended to the infinite case.

An *interpretation* (or *possible world*) w is a function from the set of propositional variables of L to the set $\{0, 1\}$, with 0 denoting falsity, and 1 denoting truth. $\mathcal{W}$ denotes the set of all possible worlds/interpretations of L. Logical entailment is denoted by $\vdash$ and logical equivalence by $\equiv$. For any set of sentences $A \subseteq L$, $[A]$ denotes the set of worlds satisfying all members of A (writing $[\phi]$ rather than $[\{\phi\}]$ for the singleton case). For a set $S \subseteq \mathcal{W}$, $Th(S)$ is the set of sentences true in all worlds in S. The object which undergoes change will be K, a consistent belief set (i.e., a deductively closed, consistent set of sentences). We take K to be arbitrary but fixed throughout. For any belief set K' and $\phi \in L$, $K' + \phi$ will denote the *expansion* of K' by ϕ, i.e., $K' + \phi = Cn(K' \cup \{\phi\})$. Given a total pre-order (i.e., a transitive, connected relation) $\leq$ on $\mathcal{W}$ and $S \subseteq \mathcal{W}$, $\min(S, \leq)$ will denote the set of $\leq$-minimal elements of S.

1.1 Removal operators

We assume that for all removal operators $\divideontimes$, $K \divideontimes \phi$ is only defined for non-tautologous propositions and refer to the set of non-tautologous members of L as L_*. The limiting case requires only a minor emendation. We make this choice for ease of technical presentation. We refer to these as *removal* operators because their use results in an epistemic input ϕ being removed from the belief set. However, as we shall see in Section 7, the extreme case where the removal of a belief ϕ results in the addition of $\neg\phi$ is included in the framework. In this paper, the following four properties will be considered as fundamental to any reasonable notion of belief removal:

(B1) $K \circledast \phi = Cn(K \circledast \phi)$
(B2) $\phi \notin K \circledast \phi$
(B3) If $\phi_1 \equiv \phi_2$ then $K \circledast \phi_1 = K \circledast \phi_2$
(B4) $K \circledast \bot = K$

Rules **(B1)–(B3)** belong to the six *basic AGM contraction postulates* [9]. Rule **(B4)** is a weakened version—under our assumption that K is consistent—of another, **(Vacuity)**.

Definition 1 *A* removal operator (for K) *is any operator satisfying* **(B1)–(B4)**.

2 BASIC REMOVAL

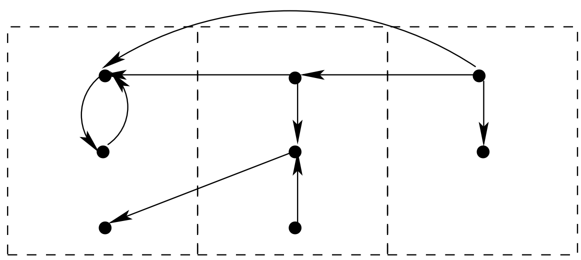
We now set up our most general semantic construction of belief change operators. Our goal is to take *full* AGM contraction as a starting point, i.e. belief contraction operators adhering to the eight AGM contraction postulates, including the two so-called *supplementary* AGM contraction postulates [9]. In line with this goal we assume a total pre-order $\leq$ *anchored* on $[K]$ i.e., $[K] = \min(\mathcal{W}, \leq)$. [1] As usual we take $\leq$ to be an ordering of plausibility on the worlds, with worlds lower down in the ordering assessed as more plausible. In what follows, $\sim$ will always denote the symmetric closure of $\leq$, i.e., $w_1 \sim w_2$ iff both $w_1 \leq w_2$ and $w_2 \leq w_1$.

In order to generalise full AGM contraction, we assume we are also given a *second* binary relation $\preceq$ on $\mathcal{W}$. The only requirement we place on $\preceq$, at least initially, is that it is a reflexive sub-relation of $\leq$. These two orderings provide the *context* in which an agent makes changes to its current beliefs. Intuitively, $\preceq$ is intended to serve as an aid to the first ordering $\leq$ in the provision of the context in which belief change should occur. This explains why $\preceq$ is required to be a sub-relation of $\leq$. In the process of providing such a context, the role of $\preceq$ will be to relate relevant worlds to one another (see Section 2.2). This justifies the requirement of $\preceq$ to be reflexive: every world is at least relevant to itself.

Definition 2 $(\leq, \preceq)$ *is a* K-context *iff* $\leq$ *is a total pre-order (on* $\mathcal{W}$*) anchored on* $[K]$*, and* $\preceq$ *is a reflexive sub-relation of* $\leq$.

The following picture shows what a K-structure looks like:

[1] But see the work by Booth et al. [15] where the assumption that $\leq$ is a total pre-order is relaxed.



The dots represent all the possible worlds in $\mathcal{W}$; the dashed rectangles represent the different $\sim$-equivalence classes, linearly ordered from left to right, with the lowest, i.e., the set $[K]$, appearing first on the left. Thus, for any two worlds w, w' appearing in the same rectangle we have $w \sim w'$, while if w appears in a rectangle strictly to the left of w' then $w < w'$. This is enough to depict $\leq$. The second ordering $\preceq$ is depicted by the arrows. An arrow from w' to w denotes $w \preceq w'$.[2] For convenience we omit all reflexive arrows. We emphasise here that $\preceq$ is *not* assumed to be transitive in general. The one real restriction is that it is not allowed to have an arrow crossing a dashed boundary from left to right. This is because $\preceq \subseteq \leq$.

Given a belief set K and a K-context $(\leq, \preceq)$, we use $(\leq, \preceq)$ to define a *removal operator* $⋇_{(\leq,\preceq)}$ for K by setting, for all $\phi \in L_*$,

$$K ⋇_{(\leq,\preceq)} \phi = Th(\{w \mid w \preceq w' \text{ for some } w' \in \min([\neg\phi], \leq)\}).$$

That is, the models of the belief set resulting from a removal of ϕ are obtained by locating all the $\leq$-best models of $\neg\phi$, and adding to those all worlds that are at least as $\preceq$-plausible.

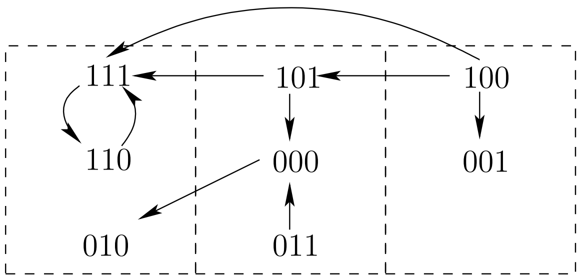
Definition 3 *$⋇$ is a* basic removal operator *(for K) iff $⋇$ is equal to $⋇_{(\leq,\preceq)}$ for some K-context $(\leq, \preceq)$.*

2.1 Examples

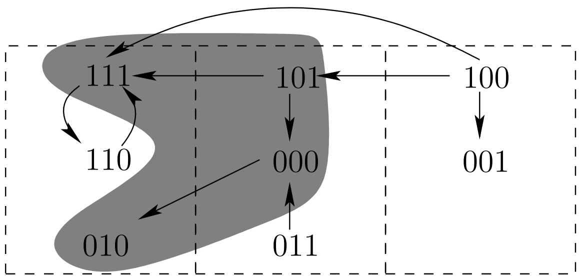
Suppose our language L contains precisely three propositional variables p, q and r. We will denote each possible world by a triple xyz of 0s or 1s, where x, y and z denote the truth-value according to that world of p, q and r respectively. So, for example, 010 denotes that world in which both p and r are false and q is true. Now suppose $K = Cn(q \wedge (r \rightarrow p))$ (so $[K] = \{111, 110, 010\}$) and let the following

[2] Observe that arrows therefore point in the direction of the worlds *lower* down in the ordering.

picture represent a particular K-context $(\leq, \preceq)$.

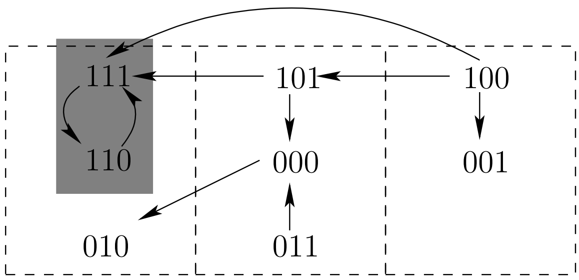


Suppose we want to remove sentence q from K. First we obtain the $\leq$-minimal models of $\neg q$. These are 101 and 000, which can be found in the second lowest $\leq$-rank. Then we add all worlds which are below these according to $\preceq$. Doing this leads us to the set of worlds covered by the grey area in the picture below:



Thus we end up with the set $\{111, 010, 101, 000\}$, i.e., $K \circledast_{(\leq,\preceq)} q = Cn(p \leftrightarrow r)$. Observe that the new model set after removal here is *not* a superset of the initial model set $[K]$. As a consequence $K \circledast_{(\leq,\preceq)} q \not\subseteq K$. Removal of q has led in this example to the aquisition of *new* beliefs (for instance $p \rightarrow r$). This shows that the widely-accepted **(Inclusion)** rule ($K \circledast \phi \subseteq K$) fails to hold in general for basic removal.

Another property for removal operators which fails is the **(Vacuity)** rule (if $\phi \notin K$ then $K \circledast \phi = K$), which says that if the sentence to be removed does not belong to the initial belief set, then its removal should leave the belief set unchanged. For suppose in the above example we want to remove $\neg r$ rather than q. Note that $\neg r \notin K$. This time the unique $\leq$-minimal model of $\neg\neg r$ is 111, appearing in the lowest level, i.e., $[K]$. The only world (other than itself) which is less than or equal according to $\preceq$ is 110:



Thus we obtain $K \circledast_{(\leq,\preceq)} \neg r = Th(\{111, 110\}) = Cn(p \wedge q)$, and we see that removal of a non-believed sentence has led to changes in the belief set, for instance the aquisition of the belief in p.

We will shortly see which properties *are* valid for basic removal, but before that we present the following result, which says that every basic removal is generated by a *unique* K-context. Thus there is a one-to-one correspondence between the K-contexts and the basic removal operators for K.

Proposition 4 *Let* $(\leq, \preceq)$ *and* $(\leq', \preceq')$ *be two* K*-contexts which are not identical. That is,* $(\leq, \preceq) \neq (\leq', \preceq')$*. Then* $\circledast_{(\leq,\preceq)} \neq \circledast_{(\leq',\preceq')}$.

Before moving on to the promised characterisation of basic removal, we demonstrate how formal K-contexts can be used to represent a particular context underlying the beliefs of an agent.

2.2 Providing Context

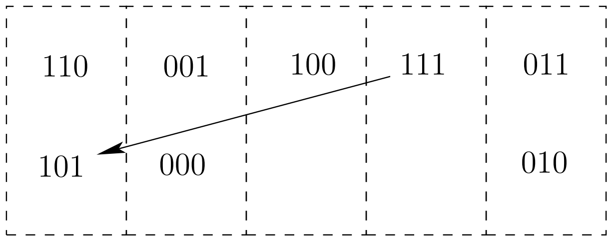
In this section we elaborate on the use of K-contexts as a way of representing the context in which an agent performs belief change by considering a simple example. In particular, we show that K-contexts can be used to represent contextual information which may be blocked by the current beliefs of an agent, but that a belief removal may trigger the unblocking of this information, depending on the appropriate context.

Consider the well-known case of representing information about Tweety. We are interested in using a K-context to capture, not only information about what an agent currently believes about Tweety, but also the *contextual* information about birds. More specifically, we require that the K-context should contain the information that an ostrich is a bird, as well as the *default* information that birds normally fly, but that ostriches normally don't fly. The information about ostriches being birds is more entrenched than the default information about birds being able to fly and ostriches not being able to fly. Moreover, given the principle of *specificity*, we take the default information about ostriches not being able to fly to be of a higher precedence than the default information of birds being able to fly.

Now suppose our agent finds itself in a situation in which it believes that Tweety is either an ostrich (and therefore a bird) which cannot fly, or that Tweety is a bird (but not an ostrich) which can fly.

We represent the beliefs of the agent in a language L containing precisely three propositional variables b (Tweety is a bird), o (Tweety is an ostrich) and f (Tweety can fly). As above, we will denote each possible world by a triple xyz of 0s or 1s, where x, y and z denote the truth-value according to that world of b, o and f

respectively. In this case we have $K = Cn(b \wedge \neg(o \leftrightarrow f))$ (so $[K] = \{110, 101\}$) and we let the following picture represent our chosen K-context $(\leq, \preceq)$.



The intuition for the choice of $(\leq, \preceq)$ can be explained as follows. The *worst* $\leq$-worlds are those in which the information that an ostrich is a bird is violated: the worlds 011 and 010 in which Tweety is an ostrich but not a bird. This is followed by the single world 111 which violates the default information about ostriches not being able to fly. Next is the single world 100 which violates the default information about birds being able to fly. After this we get the two worlds 001 and 000, both of which are compatible with all the contextual information, but are not models of the agent's current beliefs. So in these worlds Tweety is neither a bird nor an ostrich and the contextual information therefore has nothing to say about its flying abilities (or lack thereof). The *best* $\leq$-worlds are, of course, the models of K: the worlds 110 and 101.

The second ordering $\preceq$ is now used to capture a context relating to the manner in which the default assertions apply, as well as how they interact. It indicates explicitly that a world in which Tweety is a flying bird, but not an ostrich, is preferred over the world in which Tweety is a flying bird and also an ostrich, an indication that the default assertion indicating that ostriches don't fly is to be preferred over the default assertion that birds fly.

Suppose now that we want to remove the sentence that Tweety is an ostrich from the agent's beliefs. Observe that the agent currently does not believe that Tweety is an ostrich. Formally, we want to remove the sentence o from K. To do so, we first obtain the $\leq$-minimal models of $\neg o$. This is the world 101, found in the lowest $\leq$-rank (and one of the models of K). Then we add all worlds which are below these according to $\preceq$. In this case, nothing is added, and we end up with the set $\{101\}$, i.e., $K \circledast_{(\leq,\preceq)} o = Cn(b \wedge \neg o \wedge f)$. Or more informally, the agent now believes that Tweety is a bird, but not an ostrich, which can fly. So, the explicit removal of the information that Tweety is an ostrich acts as a mechanism for unblocking the default information that birds normally fly, and we end up with a belief set in which Tweety the bird is assumed not to be an ostrich, and is able to fly.

Also, it is easily checked that similar results are obtained if the *negation* of o (the information that Tweety is not an ostrich) is removed from K. In this case the default information about ostriches normally flying is unblocked, and our agent ends up believing that Tweety is an ostrich (and a bird) which is assumed not to fly.

Finally, suppose that we want to remove from the agent's beliefs the (default) assertion that Tweety being an ostrich implies that it cannot fly, an explicit assertion not currently in the agent's belief set. Formally we want to remove the sentence $o \rightarrow \neg f$ from K. To do so we first obtain the $\leq$-minimal models of $\neg(o \rightarrow \neg f)$, in this case just the single world 111. Then we to add to it all worlds which are below 111 according to $\preceq$ (in this case the world 101) and we end up with the set $\{111, 101\}$. From this it follows that $K \circledast_{(\leq,\preceq)} (o \rightarrow \neg f) = Cn(b \wedge f)$, and the agent thus ends up believing that Tweety is a flying bird. So the explicit removal of the more specific default assertion (Tweety being an ostrich implying that it cannot fly) frees up the remaining default assertion (Tweety being a bird implying that it can fly) to fire, and we end up with a belief set in which Tweety the bird is assumed to be able to fly.

To conclude this subsection, observe that belief removal in these examples satisfy neither **(Inclusion)** nor **(Vacuity)**.

2.3 Characterising Basic Removal

Basic removal is characterised by the following postulates, in addition to the fundamental rules **(B1)–(B4)**: [3]

(B5) If $\theta \in K \circledast (\theta \wedge \phi)$ then $\theta \in K \circledast (\theta \wedge \phi \wedge \psi)$
(B6) If $\theta \in K \circledast (\theta \wedge \phi)$ then $K \circledast \phi \subseteq K \circledast (\theta \wedge \phi)$
(B7) $(K \circledast \theta) \cap (K \circledast \phi) \subseteq K \circledast (\theta \wedge \phi)$
(B8) If $\phi \notin K \circledast (\theta \wedge \phi)$ then $K \circledast (\theta \wedge \phi) \subseteq K \circledast \phi$

The rules above are familiar from the belief change literature. Rules **(B7)** and **(B8)** are the two *supplementary* AGM contraction postulates [9], while **(B5)** and **(B6)** both follow from the AGM postulates (see [9,16,17]). The latter rule is closely related to the well-known rule *Cut* from non-monotonic inference [18], while the former is sometimes known in the literature as Conjunctive Trisection. A slight reformulation may be found already in [9] under the name Partial Antitony. Another reformulation of it is the following:

Proposition 5 *Let $\circledast$ be any removal operator. Then $\circledast$ satisfies* **(B5)** *iff it satisfies:*

(B5′) $K \circledast \theta \subseteq (K \circledast (\theta \wedge \phi)) + \neg\theta$

[3] The list given in [1] contained one extra rule, viz. '$K \circledast \phi \subseteq K + \neg\phi$'. It turns out this rule is derivable from the others (mainly **(B5)**). See rule **(X1)** in Lemma B in the appendix.

The remaining two basic AGM contraction rules, which are both missing from the list **(B1)–(B8)**, are **(Inclusion)** (see previous subsections) and **(Recovery)**:

(Recovery) $K \subseteq (K \circledast \phi) + \phi$

(Inclusion) has been questioned before by Bochman [19] and Booth et al. [10], the latter leading to the study of *belief liberation* operators. **(Recovery)** has been questioned in many places in the literature (e.g. [16,3]). Briefly, liberation operators cater to the intuition that removing a belief from an agent's corpus can remove the reasons for not holding others and hence lead to the inclusion of new beliefs.

Theorem 6 *Let K be a belief set and $\circledast$ an operator for K. Then $\circledast$ is a basic removal operator for K iff $\circledast$ satisfies* **(B1)–(B8)**.

Given Theorem 6, we see that basic removals seem closely related to the similarly general approach to removal presented by Bochman [19, Ch. 12]. Like basic removal, Bochman's operators in their most general form fail to validate **(Vacuity)**, **(Inclusion)** and **(Recovery)**, while they *do* satisfy **(B5)–(B7)**.

The completeness part of Theorem 6 is proved by using the following construction of a pair of orderings from a given belief set and basic removal operator.

Definition 7 *The structure $(\leq, \preceq)$ obtained from a belief set K and a basic removal operator $\circledast$, and denoted by $\mathcal{C}(K, \circledast)$ is defined as follows, for $w_1, w_2 \in \mathcal{W}$:*

$(\leq)$ $w_1 \leq w_2$ *iff* $\neg\alpha_1 \notin K \circledast (\neg\alpha_1 \wedge \neg\alpha_2)$

$(\preceq)$ $w_1 \preceq w_2$ *iff* $\neg\alpha_1 \notin K \circledast \neg\alpha_2$

where α_i is a sentence whose only model is w_i (for $i = 1, 2$).

In the theorem the structure $\mathcal{C}(K, \circledast)$ is used by checking that if $\circledast$ satisfies the postulates **(B1)–(B8)**, then $(\leq, \preceq)$ is a K-context and that $\circledast = \circledast_{(\leq,\preceq)}$. We employ this construction throughout the paper to prove that certain postulates are complete for certain sub-classes of basic removal. (See the appendix for full proofs.)

We now investigate how different requirements on the second ordering of plausibility $\preceq$ and its interplay with $\leq$ help us characterise different belief removal operations.

3 TRANSITIVE REMOVAL

The first two constraints on $\preceq$ may be viewed as necessary extra requirements on K-contexts. This is because they both lead to plausible properties of removal operators which, as we shall see later, are common to virtually all the proposed removal operators from the literature.

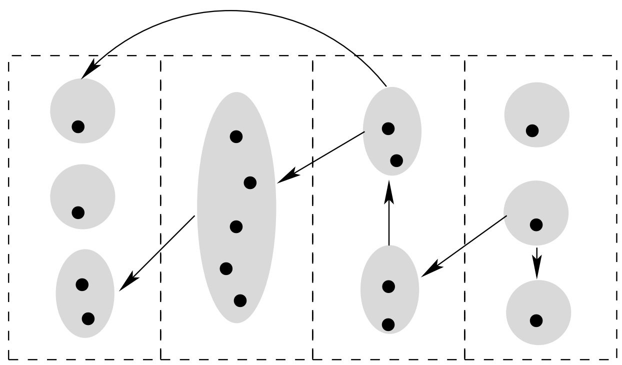
3.1 Transitivity

First we investigate the effect of setting the second order $\preceq$ to be transitive, i.e., $\preceq$ becomes a pre-order. We refer to the K-context $(\leq, \preceq)$ as transitive if $\preceq$ is transitive.

Definition 8 *We call* $\circledast$ *a* transitive removal operator *(for* K*) iff* $\circledast$ *is equal to* $\circledast_{(\leq,\preceq)}$ *for some transitive* K*-context* $(\leq, \preceq)$.

Transitive removal operators may be alternatively described as follows. As with any pre-order, the relation $\preceq$ partitions $\mathcal{W}$ into a set $\mathcal{W}/\simeq$ of equivalence classes via the relation $\simeq$ defined by $w_1 \simeq w_2$ iff both $w_1 \preceq w_2$ and $w_2 \preceq w_1$. The set $\mathcal{W}/\simeq$ is partially-ordered by the relation $\preceq^*$ defined by $[w_1]_\simeq \preceq^* [w_2]_\simeq$ iff $w_1 \preceq w_2$. Meanwhile, we can also define a relation $\leq^*$ on $\mathcal{W}/\simeq$ by $[w_1]_\simeq \leq^* [w_2]_\simeq$ iff $w_1 \leq w_2$. It is easy to check that $\leq^*$ is well-defined and that $\leq^*$ is a total pre-order on $\mathcal{W}/\simeq$ such that $\preceq^* \subseteq \leq^*$.

We can picture transitive K-contexts as follows.

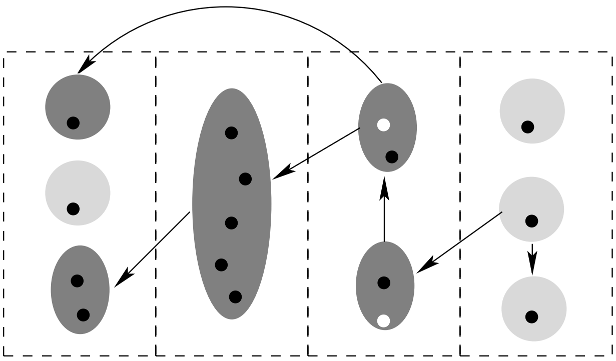


Here, the grey ellipses now represent the $\simeq$-equivalence classes from the previous paragraph. An arrow leading from one ellipse to another means that for all worlds w in the second class and all worlds w' in the first class we have $w \preceq w'$ and $w' \not\preceq w$. The full relation $\preceq$ is obtained by taking the transitive closure of the arrows.

Furthermore we have, for each $\phi \in L_*$, $K \circledast_{(\leq,\preceq)} \phi = Th(\bigcup \Upsilon)$, where

$$\Upsilon = \{X \in \mathcal{W}/\simeq \mid X \preceq^* Y \text{ for some } Y \in \min(\neg\phi, \leq^*)\},$$

and where $\min(\neg\phi, \leq^*)$ denotes the set of $\leq^*$-minimal elements $Y \in \mathcal{W}/\simeq$ such that $Y \cap [\neg\phi] \neq \emptyset$. Note how worlds belonging to the same equivalence class are ‘indistinguishable’ to the agent using the K-context $(\leq, \preceq)$.



For example, suppose the $\leq$-minimal models of $\neg\phi$ are the white dots. The new model set is then just the union of the dark grey ellipses.

The next result shows how we can axiomatically characterise the class of transitive removal operators

Proposition 9 (i). *If* $(\leq, \preceq)$ *is transitive then* $\divideontimes_{(\leq,\preceq)}$ *satisfies*[4] *:*

(BTran) *If* $K \divideontimes \theta \subseteq (K \divideontimes \phi) + \neg\phi$ *then* $K \divideontimes \theta \subseteq K \divideontimes \phi$

(ii). *If* $\divideontimes$ *is a removal operator satisfying* **(BTran)** *then the relation* $\preceq$ *of* $\mathcal{C}(K, \divideontimes)$ *is transitive.*

The reader familiar with the belief change literature will immediately recognise the right-hand-side of the antecedent of this rule as the *Levi Identity* [2], which is commonly employed to define operators of *revision* in terms of removal operators. The goal of a revision operation $K \times \phi$ is to produce a new belief which *must contain* the given sentence ϕ. Given any removal operator $\divideontimes$, let us denote by $\times = \mathbb{R}(\divideontimes)$ the operator defined from $\divideontimes$ via the Levi Identity, viz.

$$K \times \phi = (K \divideontimes \neg\phi) + \phi.$$

Thus **(BTran)** may be thought of as saying that if the act of *revising* by $\neg\phi$ produces a larger belief set than the act of *removing* θ, then so too will the act of merely *removing* ϕ.

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[4] **(BTran)** replaces the more complicated and rather unintuitive postulate which was used to characterise transitivity in [1], viz:

(BT) If $K \divideontimes \theta \not\subseteq K \divideontimes \phi$ then there exist $\psi, \lambda \in L_*$ such that $\phi \vdash \psi \vdash \lambda$ and $(K \divideontimes \theta) \cup (K \divideontimes \lambda) \vdash \phi$

Adding **(BTran)** to the list of rules for basic removal causes some redundancy in that list, as the next result shows.

Proposition 10 *Any removal operator which satisfies* **(BTran)** *and* **(B5′)** *also satisfies* **(B6)**.

So transitive removal operators may be characterised by **(B1)–(B5)**, **(B7)**, **(B8)** and **(BTran)**.

3.2 Priority

Now consider the following property of a K-context $(\leq, \preceq)$:

(a) If $w_1 \sim w_2$ and $w_1 \preceq w_2$ then $w_2 \preceq w_1$

Given the fact $\preceq \subseteq \leq$, this is easily seen to be equivalent to:

(a′) If $w_1 \prec w_2$ then $w_1 < w_2$

Thus if $w_1 \preceq w_2$ but not vice versa, then w_1 is strictly more plausible than w_2.

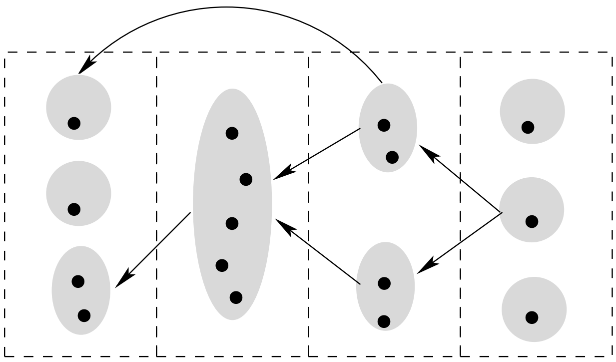
Proposition 11 (i). *If $(\leq, \preceq)$ satisfies (a) then $\divideontimes_{(\leq,\preceq)}$ satisfies:*

(BPriority) *If $\theta \in K \divideontimes \phi$ and $\phi \notin K \divideontimes \theta$ then $\theta \in K \divideontimes (\theta \wedge \phi)$*

(ii). *If $\divideontimes$ is a removal operator satisfying* **(BPriority)** *then $\mathcal{C}(K, \divideontimes)$ satisfies (a).*

The property **(BPriority)** is briefly mentioned under the name ‘Priority’ in [19], and is also briefly mentioned right at the end of [20]. It can be read as saying that if ϕ is excluded following removal of θ, but not vice versa, then θ is strictly *more entrenched* than ϕ, in the sense that when directed to exclude $\theta \wedge \phi$ (and thus to exclude at least one of θ, ϕ), θ is included and ϕ excluded.

As we indicated at the start of this section, the family of removal operators generated by transitive K-contexts satisfying (a) remains general enough to include virtually all of the families described in the rest of this paper. Thus the list of rules comprising **(B1)–(B5)**, **(B7)**, **(B8)**, **(BTran)** and **(BPriority)** can be considered a ‘common core’ of postulates for belief removal. In terms of the alternative description of transitive removal given above in terms of equivalence classes, requiring (a) of $(\leq, \preceq)$ in addition to transitivity has the effect that the relations $\leq^*$ and $\preceq^*$ on $\mathcal{W}/\simeq$ satisfy, for all $X, Y \in \mathcal{W}/\simeq$: $X \preceq^* Y$ implies $X <^* Y$ or $X = Y$, where $<^*$ is the strict part of $\leq^*$. Thus any two distinct classes X, Y which are on the same ‘level’ according to $\leq^*$ (in that both $X \leq^* Y$ and $Y \leq^* X$) are incomparable according to $\preceq^*$.



The picture for transitive K-contexts satisfying (a) is the same as that for transitive removals, except it is now disallowed to have an arrow between any two ellipses lying within the same dashed rectangle.

The next result shows how, for basic removal, the two rules **(BTran)** and **(BPriority)** may be repackaged into an equivalent single rule.

Proposition 12 (i). *If* $(\leq, \preceq)$ *is both transitive and satisfies (a) then* $\divideontimes_{(\leq,\preceq)}$ *satisfies:*

(BConserv) *If* $K \divideontimes \theta \not\subseteq K \divideontimes \phi$ *then there exists* $\lambda \in L_*$ *such that*

$\phi \vdash \lambda$ *and* $(K \divideontimes \theta) \cup (K \divideontimes \lambda) \vdash \phi$

(ii). *If* $\divideontimes$ *is a removal operator satisfying* **(BConserv)** *then* $\mathcal{C}(K, \divideontimes)$ *is transitive and satisfies (a).*

(BConserv) looks like the rules Conservativity and Weak Conservativity, which were proposed and argued-for by Hansson [21,3] and used to characterise operations of so-called *base-generated contraction*.

3.3 Strong Conservativity

By going a step further and identifying λ with ϕ in **(BConserv)** we arrive at a yet stronger postulate:

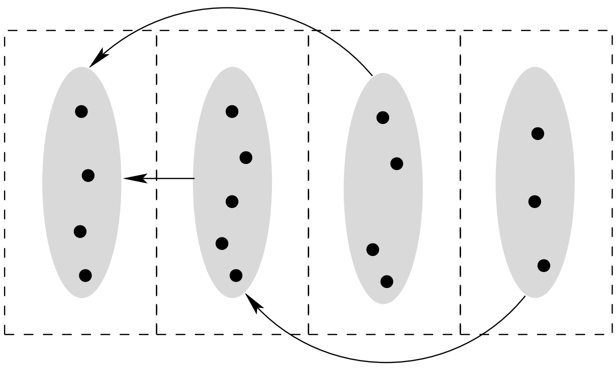
(BSConserv) If $K \divideontimes \theta \not\subseteq K \divideontimes \phi$ then $(K \divideontimes \theta) \cup (K \divideontimes \phi) \vdash \phi$

(BSConserv) is known as Strong Conservativity [21], and is used by [10] to help characterise the so-called *σ-liberation* operators (see Section 5). Booth et al. [22] also provide a detailed justification for the use of this rule. For basic removal, we can capture this property by requiring the following property, in conjunction with transitivity:

(b) If $w_1 \sim w_2$ then $w_1 \preceq w_2$

Proposition 13 (i). *If $(\leq, \preceq)$ is transitive and satisfies (b) then $\circledast_{(\leq,\preceq)}$ satisfies* **(BSConserv)**. (ii). *If $\circledast$ is a removal operator satisfying* **(BSConserv)** *then $\mathcal{C}(K, \circledast)$ is transitive and satisfies (b).*

Condition (b) implies (a). In terms of the above construction in terms of $\mathcal{W}/\simeq$, requiring $\preceq$ to be transitive while strengthening (a) to (b) has the effect that the relation $\leq^*$ becomes a *total order* on $\mathcal{W}/\simeq$.



In terms of the picture above, this means we are now restricted to just one ellipse per dashed rectangle. Note that this family will crop up later, turning out to be the family of σ-liberation operators.

4 TOWARDS AGM CONTRACTION

It was noted in Section 2 that basic removal does not satisfy the three basic AGM contraction postulates **(Vacuity)**, **(Inclusion)** and **(Recovery)**. In Section 7 it is shown that the severe withdrawal operators, which are known not to satisfy **(Recovery)** [12], are all basic removal operators, thus proving that **(Recovery)** fails for basic removal. 'One half' of **(Vacuity)**, however, *is* valid for basic removal:

Proposition 14 *Let $\circledast$ be a basic removal operator for K, then $\circledast$ satisfies: If $\phi \notin K$ then $K \subseteq K \circledast \phi$*

The 'missing half' of **(Vacuity)** is: If $\phi \notin K$ then $K \circledast \phi \subseteq K$. Clearly this rule doubles as a weakened version of **(Inclusion)**. Thus we see that, for basic removal operators, **(Inclusion)** actually implies **(Vacuity)**. In the rest of this paper we will adopt the following notational conventions for describing removal operators:

- The symbol $\circledast$ will be used to refer to members of the general family of basic removal operators, when nothing is assumed about whether the operator satisfies

(Vacuity) or **(Inclusion)**.

- The symbol $\div$ will be used if the removal operator is intended or known to satisfy **(Vacuity)**, but not necessarily **(Inclusion)**.
- Symbol $\dot{-}$ will be used if the operator is intended or known to satisfy **(Inclusion)** as well as **(Vacuity)**.

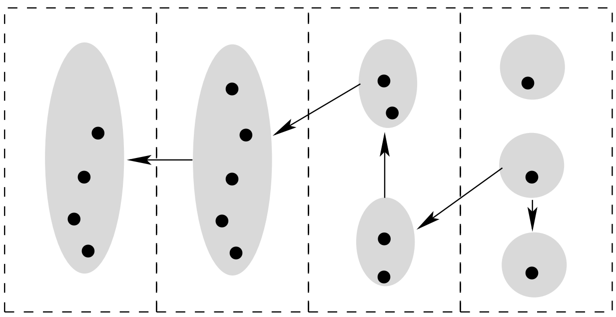
Now let us verify under what conditions on $(\leq, \preceq)$ each of these postulates are satisfied by basic removal operators.

4.1 Vacuity

To ensure that $\circledast_{(\leq,\preceq)}$ satisfy all of **(Vacuity)**, we require that all $\leq$-minimal elements (i.e., all elements of $[K]$) are $\preceq$-connected, i.e.,

(c) If $w_1, w_2 \in \min(\mathcal{W}, \leq)$ then $w_1 \preceq w_2$

In terms of the picture for *transitive* removal, this corresponds to the requirement that there is only one ellipse in the leftmost dashed-rectangle, which represents the minimal $\leq$-rank.



Proposition 15 (i). *If* $(\leq, \preceq)$ *satisfies (c) then* $\circledast_{(\leq,\preceq)}$ *satisfies* **(Vacuity)**. (ii). *If* $\circledast$ *is a removal operators satisfying* **(Vacuity)** *then* $\mathcal{C}(K, \circledast)$ *satisfies (c).*

As is easily verified, (c) is implied by condition (b). Thus we see that any basic removal satisfying **(BSConserv)** satisfies **(Vacuity)**. However, since (c) is *not* implied by (a), **(Vacuity)** is *not* valid for transitive removals satisfying (a).

Shouldn't **(Vacuity)** be a basic requirement for *any* rational removal operation? From a purely *minimal change* point of view it is certainly hard to contest, but we would nevertheless argue there *are* scenarios in which it can fail. Consider an agent with equally good reasons to believe each of p and $\neg p$. In this situation the agent remains cautious and commits to believe neither p nor $\neg p$. But if this agent were then to receive information that undermines p then it would come to believe (or assign significantly more plausibility to) $\neg p$.

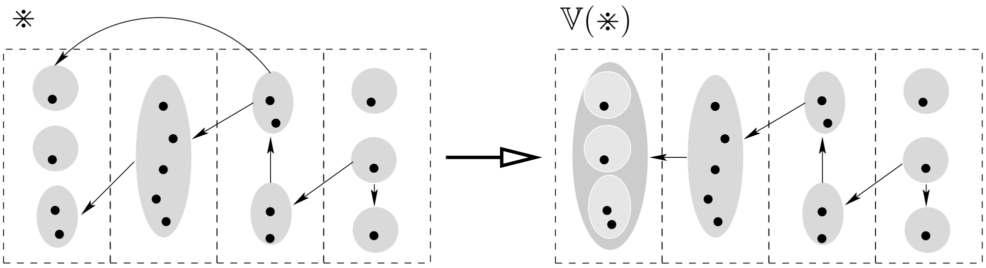
One could always try and *force* a given basic removal $\circledast$ to satisfy **(Vacuity)** by defining a new operator $\div'$ from $\circledast$ by

$$K \div' \phi = \begin{cases} K & \text{if } \phi \notin K \\ K \circledast \phi & \text{otherwise.} \end{cases}$$

It is straightforward to show that $\div'$ so defined satisfies **(B1)–(B8)**, and so again forms a basic removal.

Proposition 16 *If $(\leq, \preceq)$ is the K-context corresponding to $\circledast$, then the K-context corresponding to $\div'$ defined above is $(\leq, \preceq')$, where $\preceq'$ is obtained from $\preceq$ by setting $w_1 \preceq' w_2$ iff $w_1 \preceq w_2$ or $w_1, w_2 \in [K]$.*

However we run into difficulties in the case of *transitive* removal, for it turns out that rule **(BTran)** is *not* preserved. This is because if $\preceq$ is transitive then $\preceq'$ need not be. Indeed it is quite possible to have three models w_1, w_2, w_3 such that $w_1, w_2 \in [K]$, $w_3 \notin [K]$, $w_1 \not\preceq w_3$ and $w_2 \preceq w_3$. Then $w_1 \preceq' w_2 \preceq' w_3$ but $w_1 \not\preceq' w_3$. How can we modify a transitive removal operator so that it satisfies **(Vacuity)**? The answer is to just take the transitive closure of $\preceq'$ above. It is easy to see this is the same thing as setting $w_1 \preceq'' w_2$ iff either $w_1 \preceq w_2$ or $[w_1 \in [K]$ and $w' \preceq w_2$ for *some* $w' \in [K]]$. In terms of the picture, all we do is coalesce all the ellipses in the leftmost dashed rectangle into one, and leave all the arrows as is, so that any arrow which was previously going into *any* ellipse in this rectangle is just pointing now instead at the unique single ellipse there. (Following this step we may remove any redundant arrows.) If $\circledast$ is the basic removal operator generated by $(\leq, \preceq)$, then we will denote by $\mathbb{V}(\circledast)$ the removal operator generated by $(\leq, \preceq'')$ as defined above.



Obviously $\preceq''$ is transitive. So $\circledast_{(\leq,\preceq'')}$ is a transitive removal operator which satisfies **(Vacuity)**. Also note that if $(\leq, \preceq)$ satisfies (a), then so will $(\leq, \preceq'')$ (because $\preceq''$ does not introduce any arrows between worlds on the same level. Hence if $\circledast$ is transitive (satisfies **(BTran)**) and satisfes **(BPriority)** then so does $\mathbb{V}(\circledast)$.

The next result shows how we can express $\mathbb{V}(\circledast)$ so constructed directly in terms of $\circledast$.

Proposition 17 *Let $\circledast$ be a removal operator for K corresponding to K-context*

$(\leq, \preceq)$ *and let* $\div = \mathbb{V}(\divideontimes)$. *Then*

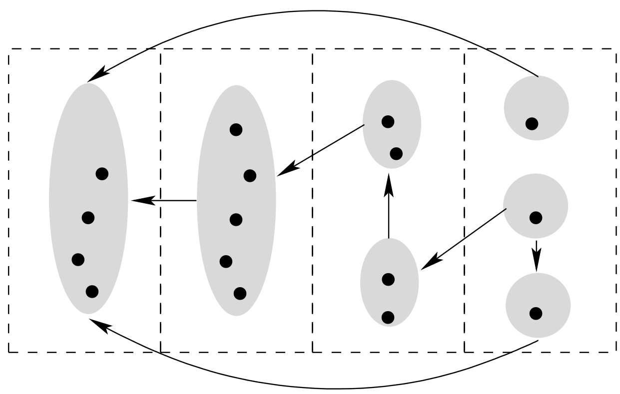
$$K \div \phi = \begin{cases} K \cap K \divideontimes \phi & \text{if } K \cup K \divideontimes \phi \text{ is consistent} \\ K \divideontimes \phi & \text{otherwise} \end{cases}$$

4.2 Inclusion

To obtain **(Inclusion)** we may add the following condition, stronger than (c):

(d) If $w_1 \in \min(\mathcal{W}, \leq)$ then $w_1 \preceq w_2$ for all w_2

So the $\leq$-minimum worlds are also the $\preceq$-minimum worlds. In the picture for *transitive* removals, this means there is only one ellipse in the leftmost dashed-rectangle, and furthermore there is a path along the arrows to this ellipse from *every* other ellipse in the picture.



Proposition 18 (i). *If* $(\leq, \preceq)$ *satisfies (d) then* $\divideontimes_{(\leq,\preceq)}$ *satisfies* **(Inclusion)**. (ii). *If* $\divideontimes$ *is a removal operator satisfying* **(Inclusion)** *then* $\mathcal{C}(K, \divideontimes)$ *satisfies (d).*

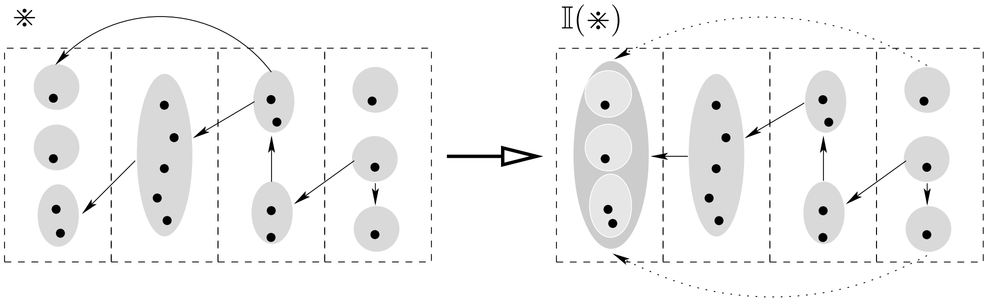
Even though basic removal operators do not satisfy **(Inclusion)** in general, it is always possible to *transform* a given basic removal $\divideontimes$ into an operator which *does* satisfy that rule. We simply take the *incarceration* $\dot{-}$ of $\divideontimes$ [10], i.e., the operator defined from $\divideontimes$ by using the following slight variant of the *Harper Identity* [2]:

$$K \dot{-} \phi = K \cap (K \divideontimes \phi).$$

We shall denote the incarceration of $\divideontimes$ by $\mathbb{I}(\divideontimes)$. It can be shown the incarceration of a basic removal operator is always itself a basic removal:

Proposition 19 *If* $(\leq, \preceq)$ *is the* K*-context corresponding to* $\divideontimes$*, then the* K*-context corresponding to* $\mathbb{I}(\divideontimes)$ *is* $(\leq, \preceq'')$*, where* $\preceq''$ *is obtained from* $\preceq$ *by setting* $w_1 \preceq'' w_2$ *iff* $w_1 \preceq w_2$ *or* $w_1 \in [K]$. *Furthermore:*

(i). *If $\preceq$ is transitive then so is $\preceq''$.*
(ii). *If $\preceq$ satisfies (a) then so does $\preceq''$.*
(iii). *If $\preceq$ satisfies (b) then so does $\preceq''$.*



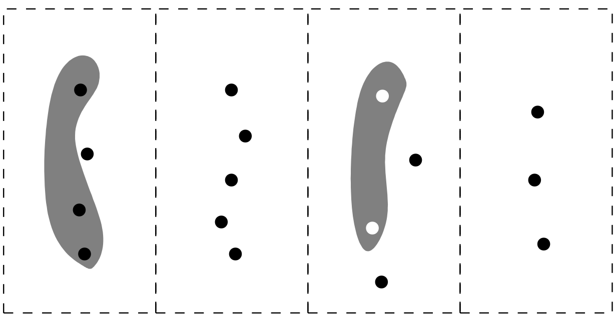
Part *(i)* of the above proposition says that the incarceration of any *transitive* removal operator is again transitive, while parts *(ii)* and *(iii)* imply that the rules **(BPriority)**, **(BConserv)** and **(BSConserv)** are each preserved under taking incarcerations.

4.3 Recovery

To obtain **(Recovery)** it suffices to require the following condition:

(e) If $w_1 \preceq w_2$ then $w_1 = w_2$ or $w_1 \in \min(\mathcal{W}, \leq)$

So, apart from itself, nothing but $\leq$-minimal worlds may be below any world in $\preceq$. This means that when removal of ϕ takes place, the new model set will consist of the $\leq$-minimal $\neg\phi$-worlds (the white dots in the picture below), together with *some subset of* the $\leq$-minimal worlds. Note that this subset may be a strict subset of $\min(\mathcal{W}, \leq)$.

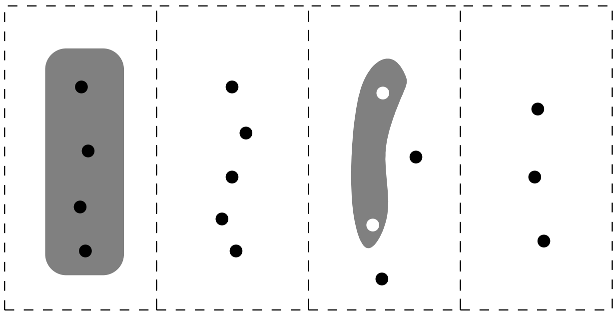


Proposition 20 (i). *If $(\leq, \preceq)$ satisfies (e) then $\circledast_{(\leq,\preceq)}$ satisfies* **(Recovery)**. (ii). *If $\circledast$ is a removal operator satisfying* **(Recovery)** *then $\mathcal{C}(K, \circledast)$ satisfies (e).*

The combination of (d) and (e) then states that the worlds below a world w in $\preceq$ are exactly w itself and the $\leq$-minimal worlds. Thus in this case $\dot{-}_{(\leq,\preceq)}$ is completely determined by $\leq$ alone via:

$$[K \dot{-}_{(\leq,\preceq)} \phi] = [K] \cup \min([\neg\phi], \leq),$$

which is precisely the AGM contraction operator generated by $\leq$ [4,5,9].



Proposition 21 *The following are equivalent:*
(i). $\dot{-}$ *is a full AGM contraction operator (i.e., satisfying the basic and supplementary AGM postulates).*
(ii). $\dot{-}$ *satisfies* **(B1)–(B8)** *plus* **(Inclusion)** *and* **(Recovery)**.
(iii). $\dot{-} = \dot{-}_{(\leq,\preceq)}$ *for some* $(\leq, \preceq)$ *which satisfies (d) and (e).*

Observe that since (d)+(e) implies transitivity and (a), every full AGM contraction satisfies **(BTran)**, **(BPriority)** and **(BConserv)**.

5 BELIEF LIBERATION

Booth et al. [10] present two models of belief liberation operators, each in terms of finite sequences of sentences. The second model, *linear liberation*, is more general than the first, σ*-liberation* as the class of liberation operators it generates includes those generated by the first. The first construction employs a linearly ordered sequence of sentences and the second a set of candidate belief sets one of which corresponds to the agent's set after belief retraction. They also provide axiomatic characterisations of each of these classes.

A K-sequence is any sequence of sentences $\rho = (\beta_1, \ldots, \beta_m)$ such that $K = Cn(\beta_1)$. For any K-sequence ρ we can define a removal operator $\div_\rho$ by setting [5]

$$K \div_\rho \phi = \begin{cases} Cn(\beta_i) \text{ where } i = \min\{k \mid \beta_k \nvdash \phi\} & \text{if } \bigvee_k \beta_k \nvdash \phi \\ Cn(\emptyset) & \text{otherwise} \end{cases}$$

Then an operator $\div$ for K is a linear liberation operator (for K) iff $\div = \div_\rho$ for some K-sequence ρ.

Linear liberation is characterised by **(B1)–(B3)** plus **(Vacuity)** and the following rule: [6]

(Hyperreg) If $\theta \notin K \div (\theta \wedge \phi)$ then $K \div (\theta \wedge \phi) = K \div \theta$

This is the rule termed Hyperregularity in [21]. The first thing to note about **(Hyperreg)** is that, in the presence of **(B1)–(B4)**, it actually implies **(Vacuity)** *and* the remaining rules for basic removal **(B5)–(B8)**. Thus we see:

Proposition 22 *$\div$ is a linear liberation operator iff it is a basic removal operator which satisfies* **(Hyperreg)**.

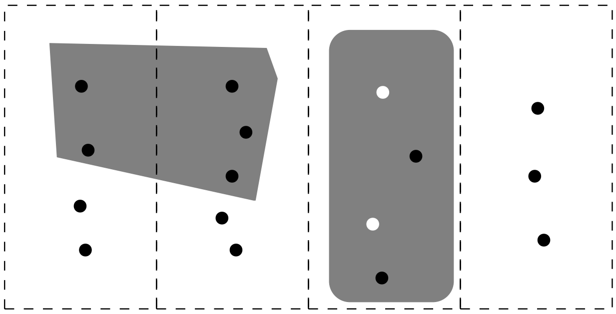
Is there a condition on $(\leq, \preceq)$ which corresponds exactly to **(Hyperreg)**? It turns out the following condition does the trick:

(f) If $w_1 \sim w_2$ and $w_3 \preceq w_1$ then $w_3 \preceq w_2$

Rule (f) says that whether or not a world w_3 is below w_1 according to $\preceq$ depends only on the $\leq$-plausibility rank of w_1. In terms of the picture, a major effect of this is that the new model set when removing ϕ will *always* contain *all* worlds which are in the same $\leq$-rank as the $\leq$-minimal $\neg\phi$-worlds. Furthermore the set of models below this rank which are to be included in the new model set is determined entirely by this rank. Thus the number of possible distinct belief sets which may result from

[5] Booth et al. [10] also allowed the removal of tautologies, a difference that may safely be ignored.
[6] Taking into account that here, unlike in [10], we don't allow the removal of a tautology.

an operation of removal is exactly the number of plausibility ranks.



Proposition 23 (i). *If* $(\leq, \preceq)$ *satisfies (f) then* $\div_{(\leq,\preceq)}$ *satisfies* **(Hyperreg)**. (ii). *If* $\div$ *is a removal operator satisfying* **(Hyperreg)** *then* $\mathcal{C}(K, \circledast)$ *satisfies (f).*

Thus we see linear liberation operators may be represented by the class of K-contexts which satisfy (f).

Note that (f) doesn't imply transitivity, but does imply (b) (and therefore (a)). In the presence of **(B1)** and **(B2)**, **(Hyperreg)** implies the following rule known as Decomposition:

$$K \circledast (\theta \wedge \phi) = K \circledast \theta \text{ or } K \circledast (\theta \wedge \phi) = K \circledast \phi$$

As is noted in [2, p66], this condition is not desirable in general. For this reason (f) might be too strong to be a general requirement.

5.2 σ-liberation

The definition of σ-liberation is, like linear liberation, based on sequences $\sigma = (\alpha_1, \ldots, \alpha_n)$ of sentences, although the sequences are used in a different way. One natural way to interpret the sequence is as the list of previous revision inputs the agent has received, with α_1 being the first and α_n the most recent. The construction begins by inductively defining, for each $\phi \in L$, an increasing sequence of sets of sentences $\Gamma_i(\sigma, \phi)$ by setting $\Gamma_0(\sigma, \phi) = \emptyset$ and then, for each $i = 0, \ldots, n-1$,

$$\Gamma_{i+1}(\sigma, \phi) = \begin{cases} \Gamma_i \cup \{\alpha_{n-i}\} & \text{if } \Gamma_i \cup \{\alpha_{n-i}\} \not\vdash \phi \\ \Gamma_i & \text{otherwise.} \end{cases}$$

That is, starting at the end with α_n, we work our way backwards through the sequence, adding each sentence as we go, provided adding it to the sentences collected up to that point does not lead to the inference of ϕ. If $Cn(\Gamma_n(\sigma, \bot)) = K$

then we say σ is a belief sequence *relative to* K. Then every belief sequence relative to K defines a removal operator $\div_\sigma$ by setting

$$K \div_\sigma \phi = Cn(\Gamma_n(\sigma, \phi)).$$

Finally, an operator $\div$ for K is a σ-liberation operator (for K) iff $\div = \div_\sigma$ for some belief sequence σ relative to K.

Booth et al. [10] show that the σ-liberation operators are precisely those linear liberation operators which satisfy **(BSConserv)**. Using this fact together with Propositions 13 and 23 allows us to deduce:

Proposition 24 *$\div$ is a σ-liberation operator iff $\div$ is equal to $\div_{(\leq,\preceq)}$ for some transitive $(\leq, \preceq)$ satisfying (b) and (f).*

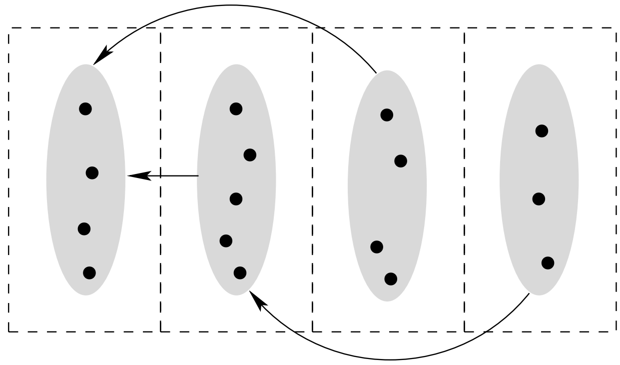
However we can simplify here, for as soon as $\preceq$ is transitive, conditions (b) and (f) become *equivalent*:

Proposition 25 *Let $(\leq, \preceq)$ be a transitive K-context. Then $(\leq, \preceq)$ satisfies (b) iff $(\leq, \preceq)$ satisfies (f).*

This means that in Proposition 24 it is unnecessary to require both (b) *and* (f) – just one of them will suffice. Depending on which one we choose to retain, we obtain two different characterisations of σ-liberation which provide alternatives to the one from [10]:

Proposition 26 *The following are equivalent:*
(i). $\div$ *is a σ-liberation operator.*
(ii). $\div$ *is a linear liberation operator which satisfies* **(BTran)**.
(iii). $\div$ *is a basic removal operator which satisfies* **(BSConserv)**.

The equivalence *(i)⇔(ii)* comes from combining Proposition 24 (retaining just (f)) with Propositions 9 and 23, while *(i)⇔(iii)* comes from combining Proposition 24 (retaining just (b)) with Proposition 13. Surprisingly, *(i)⇔(ii)* says that, in the axiomatisation of σ-liberation in [10], **(BSConserv)** may be replaced by the seemingly much weaker **(BTran)**. Meanwhile, since *(i)⇔(iii)*, σ-liberation operators inherit the nice description in terms of $\mathcal{W}/\simeq$ given for the basic removals which satisfy **(BSConserv)** at the end of Section 3 (where $\leq^*$ is a total order on $\mathcal{W}/\simeq$).



Similar characterisations for sub-classes of liberation, such as the class of *dichotomous* liberation operators [10], exist. We consider these next.

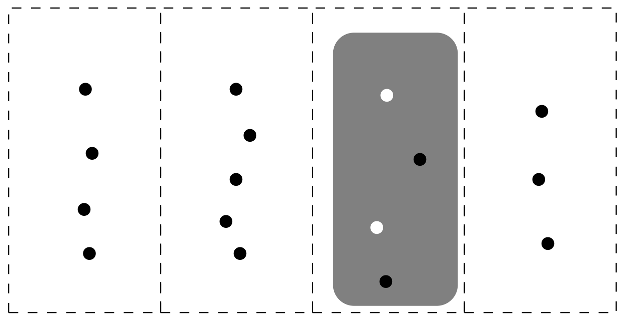
In [10] the sub-class of σ-liberation operators known as the *dichotomous* liberation operators are characterised by adding the following postulate:

(Dichotomy) $(K \circledast \theta) \cup K \circledast \phi) \nvdash \bot$ implies $K \circledast \theta = K \circledast \phi$

The following condition on $(\leq, \preceq)$ corresponds to **(Dichotomy)**:

(e′) $w_1 \preceq w_2$ iff $w_1 \sim w_2$

This condition requires that the worlds below a world w in $\preceq$ are precisely those with the same plausibility ranking as w.



Proposition 27 (i). *If* $(\leq, \preceq)$ *satisfies (e′) then* $\circledast_{(\leq,\preceq)}$ *satisfies* **(Dichotomy)**. (ii). *If* $\circledast$ *is a removal operator satisfying* **(Dichotomy)** *then* $\mathcal{C}(K, \circledast)$ *satisfies (e′).*

It turns out that adding **(Dichotomy)** to the postulates for basic removal gives exactly dichotomous liberation.

Proposition 28 *The following are equivalent:*
(i). $\circledast$ *is a dichotomous liberation operator.*
(ii). $\circledast$ *satisfies* **(B1)–(B8)** *plus* **(Dichotomy)**.

(iii). $\circledast = \circledast_{(\leq,\preceq)}$ *for some* $(\leq,\preceq)$ *which satisfies (e′).*

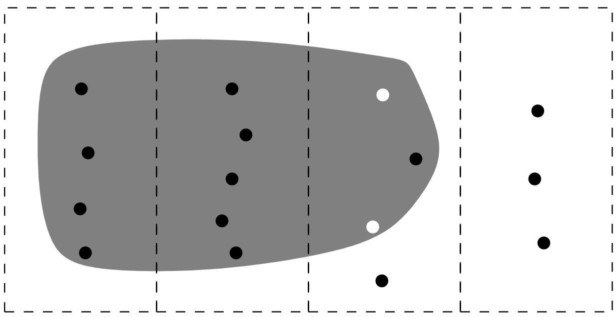
The equivalence *(i)⇔(iii)* comes from combining Proposition 24 and Proposition 27, while *(ii)⇔(iii)* comes from combining Theorem 6 and Proposition 27.

6 SYSTEMATIC WITHDRAWAL

An interesting sub-class of basic removal operators, which includes both systematic [11] and severe withdrawal [12] (see below) is obtained by requiring the following condition on $(\leq,\preceq)$:

(g) If $w_1 < w_2$ then $w_1 \preceq w_2$

where $<$ is the strict part of $\leq$. When removing ϕ, the effect is that the new model set will contain, along with the $\leq$-minimal $\neg\phi$-worlds, *all* worlds considered strictly more $\leq$-plausible, together with possibly *some* of the ϕ-worlds appearing in the same $\leq$-plausibility rank as these $\leq$-minimal $\neg\phi$-worlds.



Proposition 29 (i). *If* $(\leq,\preceq)$ *satisfies (g) then* $\circledast_{(\leq,\preceq)}$ *satisfies:*

(B9) *If* $\theta \in K \circledast (\theta \wedge \phi)$ *then* $\phi \notin K \circledast \theta$

(ii). *If* $\circledast$ *is a removal operator satisfying* **(B9)** *then* $\mathcal{C}(K,\circledast)$ *satisfies (g).*

The class of basic removal operators $\circledast_{(\leq,\preceq)}$ such that $(\leq,\preceq)$ satisfies (g) still do not generally satisfy **(Inclusion)** or **(Vacuity)**, since condition (g) does not rule out that some $\leq$-minimal elements may be $\preceq$-unconnected. However they do come *very* close to satisfying **(Inclusion)**, in that the following is satisfied:

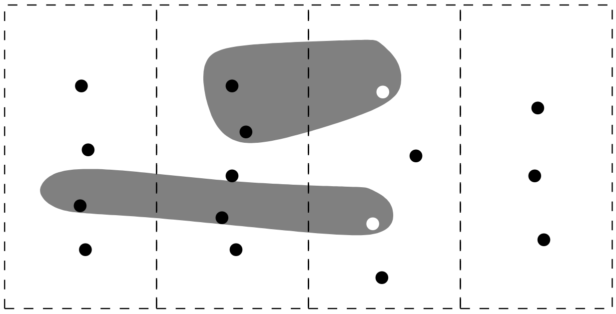
If $\theta \in K$ then $K \circledast \theta \subseteq K$

Using this fact, we see that for *this* class of operators, **(Inclusion)** and **(Vacuity)** are equivalent.

The next condition on K-contexts is, essentially, a requirement for antisymmetry to hold:

(h) If $w_1 \preceq w_2$ then either $w_1 < w_2$ or $w_1 = w_2$

So now, the ϕ-worlds appearing in the new model set after removing ϕ are selected exclusively among those considered strictly more plausible than the $\leq$-minimal $\neg\phi$-worlds.



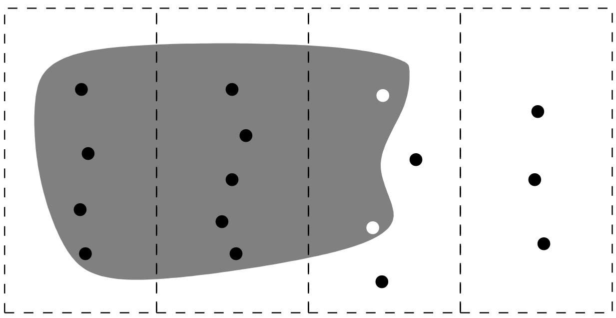
Proposition 30 (i). *If* $(\leq, \preceq)$ *satisfies (h) then* $\circledast_{(\leq,\preceq)}$ *satisfies:*

(B10) *If* $\vdash (\theta \vee \phi)$ *and* $\theta \notin K \circledast \phi$ *then* $\phi \in K \circledast (\theta \wedge \phi)$

(ii). *If* $\circledast$ *is a removal operator satisfying* **(B10)** *then* $\mathcal{C}(K, \circledast)$ *satisfies (h).*

Clearly, by requiring (h) in combination with (g) (and reflexivity) we specify $\preceq$ uniquely:

(g)+(h) $w_1 \preceq w_2$ iff either $w_1 < w_2$ or $w_1 = w_2$



Note that $\preceq$ so defined will automatically be transitive and will satisfy the condition (a) from Section 3.

Putting together Propositions 29 and 30, we have that the class of basic removal operators $\circledast_{(\leq,\preceq)}$ where $\preceq$ is defined via (g)+(h) may be axiomatically characterised by **(B1)–(B10)**. This looks very much like the class of systematic withdrawals. A systematic withdrawal operator $\dot{-}$ can be defined in terms of $\leq$ as follows [11]:

$$K \dot{-} \phi = K \cap Th(\nabla_{\leq}(\min([\neg\phi], \leq)))$$

where $\nabla_{\leq}(X) = \{v \mid \exists w \in X \text{ s.t. } v = w \text{ or } v < w\}$. Unlike systematic withdrawal, the class of removal operators defined by **(B1)–(B10)** *fails* to satisfy **(Inclusion)**/**(Vacuity)**, since all the $\leq$-minimal elements are necessarily *unconnected* according to $\preceq$. So in fact **(Vacuity)** will fail as soon as there is more than one $\leq$-minimal element. These operators satisfy instead:

If $\phi \notin K$ then $\neg\phi \in K \circledast \phi$

That is, for these operators, we see $K \circledast \phi$ is an operation which 'demotes' the status of ϕ: if its current status is 'accepted', i.e., $\phi \in K$, then its status is 'demoted' to 'undecided' i.e., $\phi, \neg\phi \notin K \circledast \phi$, while if its current status is 'undecided' then its status is 'demoted' to 'rejected'. If its status is already 'rejected' then no change occurs. However, if we take the incarcerations of these operators then we end up with precisely the class of systematic withdrawal operators.

Systematic withdrawal can also be obtained by weakening (h):

(j) If $w_1 \preceq w_2$ then $w_1 < w_2$, $w_1 = w_2$, or $w_1 \leq w' \ \forall w'$

So, unlike (h), (j) allows the models of K to be connected according to $\preceq$, although it does not force them to be.

Proposition 31 (i). *If* $(\leq, \preceq)$ *satisfies (j) then* $\circledast_{(\leq,\preceq)}$ *satisfies:*

(B11) *If* $\vdash (\theta \vee \phi)$ *and* $\theta \in K \setminus K \circledast \phi$ *then* $\phi \in K \circledast (\theta \wedge \phi)$
(ii). *If* $\circledast$ *is a removal operator satisfying* **(B11)** *then* $\mathcal{C}(K, \circledast)$ *satisfies (j).*

Since the operators obtained from (g) and (h) form a sub-class of the operators obtained from (g) and (j), the latter class still does not satisfy **(Vacuity)**. But adding (c) (and therefore **(Vacuity)**) to (g) and (j) leads exactly to systematic withdrawal.

Proposition 32 *The following are equivalent:*
(i). $\dot{-}$ *is a systematic withdrawal.*
(ii). $\dot{-}$ *satisfies* **(B1)–(B8)** *plus* **(Vacuity)**, **(B9)** *and* **(B11)**.
(iii). $\dot{-} = \circledast_{(\leq,\preceq)}$ *for some* $(\leq, \preceq)$ *which satisfies (c), (g) and (j).*

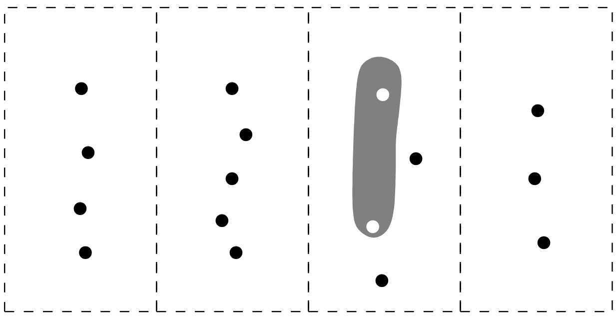
As we shall see in the next section, the class of severe withdrawals can be isolated in a similar manner.

7 LIMITING CASES

We have seen that the addition of the second ordering $\preceq$ provides us with considerable flexibility when defining removal operators. But what happens when we focus on the limits imposed on $\preceq$? In this section we consider the two cases where $\preceq$ is the *smallest* and the *largest* reflexive sub-relation of $\leq$.

7.1 AGM revision

If we take $\preceq$ to be the smallest $\preceq$, the equality relation, then the operator $\divideontimes_{(\leq,\preceq)}$ reduces to $K \divideontimes_{(\leq,\preceq)} \phi = Th(\min([\neg\phi], \leq))$,



We have the following result.

Proposition 33 (i). *If $\preceq$ is the equality relation then $\divideontimes_{(\leq,\preceq)}$ satisfies:*

(B12) $\neg\phi \in K \divideontimes \phi$.

(ii). *If $\divideontimes$ is a removal operator satisfying* **(B12)** *then $\preceq$ in $\mathcal{C}(K, \divideontimes)$ is the equality relation.*

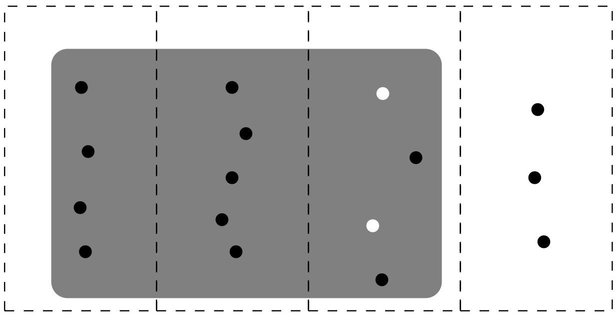
Thus we see that removing ϕ here amounts to a *revision* by its negation, and in fact that $\divideontimes_{(\leq,\preceq)}$ essentially reduces to an AGM revision function (satisfying the full list of AGM revision postulates [9]). More precisely the operator $*_{(\leq,\preceq)}$ for K defined by $K *_{(\leq,\preceq)} \phi = K \divideontimes_{(\leq,\preceq)} \neg\phi$ is an AGM revision operator. Moreover, *every* AGM revision operator can be obtained in this way. Note that in the above case, since $\phi \in K \divideontimes_{(\leq,\preceq)} \neg\phi$, the right-hand side here is equal to $(K \divideontimes_{(\leq,\preceq)} \neg\phi) + \phi$. Thus what we have is just the Levi Identity [2]. In fact a result more general holds. Recall that for any removal operator $\divideontimes$, we use $\mathbb{R}(\divideontimes)$ to denote the operator derived from $\divideontimes$ via the Levi Identity:

Proposition 34 *If $\divideontimes$ is a basic removal operator then $\mathbb{R}(\divideontimes)$ is an AGM revision operator.*

7.2 Severe withdrawal

By taking $\preceq$ to be the largest reflexive sub-relation of $\leq$ we get the full relation $\leq$, and the operator $\circledast_{(\leq,\preceq)}$ reduces to:

$$K \dot{-}_{(\leq,\preceq)} \phi = Th(\{w \mid w \leq w' \text{ for some } w' \in \min([\neg\phi], \leq)\}).$$



Thus, from the characterisation of severe withdrawal in terms of total pre-orders found in [12], we see that setting $\preceq$ equal to $\leq$ gives us the class of severe withdrawal operators. Note that $\preceq$ so defined will be transitive and satisfy condition (b) from Section 3 (and hence also (f) – see Proposition 25). From the results above it turns out we can give an axiomatic characterisation of severe withdrawal different to the ones found in the literature (see [12]). To do this note the following:

Proposition 35 *Let $(\leq, \preceq)$ be a K-context. Then $\preceq$ is equal to $\leq$ iff both (f) and (g) are satisfied.*

Using this fact with Propositions 23 and 29 then yields:

Proposition 36 $\dot{-}$ *is a severe withdrawal operator iff it satisfies* **(B1)–(B4)**, **(Hyperreg)** *and* **(B9)**.

8 RELATED WORK

It has long been recognised that extra-logical information is needed for a sufficiently general theory of belief change. Indeed, the use of plausibility orderings $\leq$ on their own to define belief contraction is testament to that realisation. Hansson [3] was probably the first to point out that, even when we are concerned with belief sets, and not belief bases, it is useful to draw a distinction between *basic* and *derived* beliefs. His unification of belief base contraction and belief set contraction [21] provides a framework for doing so, but his construction methods use belief bases, are not based on plausibility orderings on worlds, and are therefore quite different from what we propose here. It is, of course, possible to compare his approach and ours on the abstract level of postulates, but doing so is not a trivial matter, and is beyond the scope of this paper.

Cantwell [6] points out that some beliefs do not fit neatly into Hansson's categories of basic vs. derived beliefs. In addition to an *entrenchment ordering* on sentences, he also proposes an *eligibility relation* on sentences. His entrenchment orderings can be converted to our plausibility orderings $\leq$ on worlds, and he employs the entrenchment orderings in such a way that it corresponds to Rott and Pagnucco's severe withdrawal [12]. His eligibility relation regards two sentences ψ and ϕ as related iff ψ is eligible for removal whenever ϕ has to be removed. He then uses these eligibility relations to restrict the sentences removed by a given severe withdrawal (obtained from a specific entrenchment ordering).

Cantwell's work is thus similar in spirit to ours in the sense that he introduces, as part of the required extra-logical information to perform belief change, a second ordering on top of the standard entrenchment ordering. In terms of construction, it is quite different from our work, though. At present, there does not seem to be a link between his eligibility ordering on sentences and our second ordering $\preceq$ on worlds. When comparing the operators generated by our K-contexts with those generated by Cantwell's eligible contraction, it is unclear what the intersection of these two classes of removal operators looks like, except for the fact that severe withdrawal and AGM contraction are special cases of both classes. As is the case with Hansson's work, a detailed comparison between our work and that of Cantwell is possible on the abstract level of postulates. But because the construction methods are so different, this is a non-trivial task, and is beyond the scope of this paper.

A different approach to the provision of extra-logical information to characterise belief change is that of Bochman's [19] general theory of the contraction of epistemic states, which aims to unify classical belief base contraction and belief set contraction. Bochman defines an epistemic state as an ordered collection of belief sets. The contraction of an epistemic state by a sentence ϕ is defined as the intersection of the minimal belief sets which do not entail ϕ. So Bochman's method of constructing contraction operators is quite different from our method for constructing belief removal operators based on K-contexts. On the other hand, on an abstract level his approach is perhaps the closest to our notion of basic removal. Specifically, Bochman's general form of contraction satisfies **(B1)** and **(B3)–(B7)** and, as is the case for basic removal, does not satisfy **(Vacuity)**, **(Inclusion)** and **(Recovery)**.

9 CONCLUSION

In this study we have presented a unified framework for belief removal in terms of a possible world semantics which is distinctive in that it uses a pair of orderings over the set of possible worlds. We argued for the intuitive plausibility of this pair and showed how a large class of belief removal operators such as liberation, systematic and severe withdrawal operators could be characterised by using them to guide belief change. This approach makes possible the identification of hitherto unstudied sub-classes of basic removal operators, such as those obtained by requiring of $\preceq$ to be a total pre-order and a partial order. An obvious generalisation to consider in future work is the extension to propositional languages with a countably infinite number of propositional variables. Gabbay and Schlechta [14] have addressed this case, but in relation to the initial work [1] on which this paper is based.

Also, a detailed study of the connection between basic removal, base-generated contraction, and sequence-based retraction is of interest. Finally, as in any formalism for belief change, we need to consider iterated removal and how this affects the adjustment of worlds in both $\leq$ and $\preceq$, as well as the interplay between $\leq$ and $\preceq$.

Acknowledgements

David Makinson gave us some help with the title.

A Proofs for Section 2

In the proofs contained in this appendix we will sometimes treat a propositional world w as a sentence and write, e.g., $\neg w$, $w_1 \vee w_2$ etc. Whenever a world w appears in the scope of a propositional connective like this, it should be understood as standing for any sentence α such that $[\alpha] = \{w\}$. Such a sentence always exists under our assumption that L is finitely generated. The finiteness also implies that for every deductively closed set of sentences K there exists a single sentence β such that $K = Cn(\beta)$, and another useful fact that will be repeatedly used is that for any two deductively closed sets K_1, K_2 we have $K_1 \subseteq K_2$ iff $[K_2] \subseteq [K_1]$.

Proposition 4 Let $(\leq, \preceq)$ and $(\leq', \preceq')$ be two K-contexts that are not identical. That is, $(\leq, \preceq) \neq (\leq', \preceq')$. Then $\divideontimes_{(\leq,\preceq)} \neq \divideontimes_{(\leq',\preceq')}$.

PROOF. For this proof we will denote $\divideontimes_{(\leq,\preceq)}$ by just $\divideontimes$ and $\divideontimes_{(\leq',\preceq')}$ by $\divideontimes'$.

Suppose first of all that $\leq \neq \leq'$. Then, without loss, let w_1, $w_2 \in \mathcal{W}$ be such that $w_1 \leq w_2$ but $w_2 <' w_1$. Now consider the result of removing sentence $\neg(w_1 \vee w_2)$ from K using each of $\circledast$ and $\circledast'$. We will show that $\neg w_1 \notin K \circledast \neg(w_1 \vee w_2)$ but $\neg w_1 \in K \circledast' \neg(w_1 \vee w_2)$, thus proving $\circledast \neq \circledast'$ in this case. By definition we know

$$[K \circledast \neg(w_1 \vee w_2)] = \{w \mid w \preceq w' \text{ for some } w' \in \min([(w_1 \vee w_2)], \leq)\},$$

and similarly for $\circledast'$, replacing $\leq, \preceq$ by $\leq', \preceq'$. Since $[(w_1 \vee w_2)] = \{w_1, w_2\}$ and $w_1 \leq w_2$ we know $w_1 \in \min([(w_1 \vee w_2)], \leq)$ and so by reflexivity of $\preceq$ we know $w_1 \in [K \circledast \neg(w_1 \vee w_2)]$. Thus $\neg w_1 \notin K \circledast \neg(w_1 \vee w_2)$ as claimed. Meanwhile for $\circledast'$, if it were the case that $w_1 \in [K \circledast \neg(w_1 \vee w_2)]$ then we would have $w_1 \preceq' w'$ for some $w' \in \min([(w_1 \vee w_2)], \leq')$. Since $w_2 <' w_1$, the only element in $\min([(w_1 \vee w_2)], \leq)$ is w_2. Hence this would mean $w_1 \preceq' w_2$. Since $\preceq' \subseteq \leq'$ by definition of K-context this would imply $w_1 \leq' w_2$ – contradiction. Hence $w_1 \notin [K \circledast \neg(w_1 \vee w_2)]$, i.e., $\neg w_1 \in K \circledast' \neg(w_1 \vee w_2)$ as required.

Now suppose $\preceq \neq \preceq'$. Without loss now let w_1, w_2 be such that $w_1 \preceq w_2$ but $w_1 \not\preceq' w_2$. In this case we can show $\circledast$ and $\circledast'$ yield different results when removing $\neg w_2$. First note that since $[w_2] = \{w_2\}$ then $\min([w_2], \leq) = \min([w_2], \leq') = \{w_2\}$. Since $w_1 \preceq w_2$ and $w_1 \not\preceq' w_2$ we thus know $w_1 \preceq w'$ for some $w' \in \min([w_2], \leq)$ and $w_1 \not\preceq' w'$ for all $w' \in \min([w_2], \leq')$. These two in turn imply $w_1 \in [K \circledast \neg w_2]$ and $w_1 \notin [K \circledast' \neg w_2]$, hence $\neg w_1 \notin K \circledast \neg w_2$ while $\neg w_1 \in K \circledast' \neg w_2$, so we have shown $\circledast \neq \circledast'$.

Now for many of our proofs the following property, showing how the $\leq$-minimal elements of $[\neg\phi]$ can be described directly in terms of $\circledast_{(\leq,\preceq)}$, will be key:

Lemma A Let $(\leq, \preceq)$ be a K-context. The for any $\phi \in L_*$, $\min([\neg\phi], \leq) = [K \circledast_{(\leq,\preceq)} \phi)] \cap [\neg\phi]$.

PROOF. For the left-to-right inclusion we obviously have $\min([\neg\phi], \leq) \subseteq [\neg\phi]$, while $\min([\neg\phi], \leq) \subseteq [K \circledast_{(\leq,\preceq)} \phi]$ follows from the reflexivity of $\preceq$.

For the converse inclusion suppose $w \in [K \circledast_{(\leq,\preceq)} \phi)] \cap [\neg\phi]$. Then from $w \in [K \circledast_{(\leq,\preceq)} \phi)]$ we know $w \preceq w'$ for some $w' \in \min([\neg\phi], \leq)$. Since $\preceq \subseteq \leq$ we have $w \leq w'$, and so since $w \in [\neg\phi]$ and from the minimality of w' we must have $w \in \min([\neg\phi], \leq)$ as required.

Recall postulates **(B1)–(B8)**:

(B1) $K \circledast \phi = Cn(K \circledast \phi)$
(B2) $\phi \notin K \circledast \phi$

(B3) If $\phi_1 \equiv \phi_2$ then $K \circledast \phi_1 = K \circledast \phi_2$
(B4) $K \circledast \bot = K$
(B5) If $\theta \in K \circledast (\theta \wedge \phi)$ then $\theta \in K \circledast (\theta \wedge \phi \wedge \psi)$
(B6) If $\theta \in K \circledast (\theta \wedge \phi)$ then $K \circledast \phi \subseteq K \circledast (\theta \wedge \phi)$
(B7) $(K \circledast \theta) \cap (K \circledast \phi) \subseteq K \circledast (\theta \wedge \phi)$
(B8) If $\phi \notin K \circledast (\theta \wedge \phi)$ then $K \circledast (\theta \wedge \phi) \subseteq K \circledast \phi$

Note that when using the above postulates in proofs we will sometimes not explictly mention more obvious uses of some of the more fundamental postulates, e.g., by **(B3)** we know always $K \circledast (\theta \wedge \phi) = K \circledast (\phi \wedge \theta)$.

Proposition 5 Let $\circledast$ be any removal operator. Then $\circledast$ satisfies **(B5)** iff it satisfies:

(B5′) $K \circledast \theta \subseteq (K \circledast (\theta \wedge \phi)) + \neg\theta$

PROOF. To show **(B5)** implies **(B5′)**, first let $\circledast(\theta)$ be any sentence such that $K \circledast \theta = Cn(\circledast(\theta))$. Now, by **(B1)** we know $(\neg\theta \rightarrow \circledast(\theta)) \in K \circledast \theta$. Since $\theta \equiv (\neg\theta \rightarrow \circledast(\theta)) \wedge \theta$ this means $K \circledast \theta = K \circledast ((\neg\theta \rightarrow \circledast(\theta)) \wedge \theta)$ by **(B3)** and so we obtain $(\neg\theta \rightarrow \circledast(\theta)) \in K \circledast ((\neg\theta \rightarrow \circledast(\theta)) \wedge \theta)$. Applying **(B5)** to this we may deduce $(\neg\theta \rightarrow \circledast(\theta)) \in K \circledast ((\neg\theta \rightarrow \circledast(\theta)) \wedge \theta \wedge \phi)$. But $(\neg\theta \rightarrow \circledast(\theta)) \wedge \theta \wedge \phi \equiv \phi \wedge \theta$. Hence by **(B3)** we get $(\neg\theta \rightarrow \circledast(\theta)) \in K \circledast (\phi \wedge \theta)$, equivalently $K \circledast \theta \subseteq (K \circledast (\phi \wedge \theta)) + \neg\theta$ as required.

To show **(B5′)** implies **(B5)**, suppose $\theta \in K \circledast (\theta \wedge \phi)$. Now, **(B5′)** (with a little help from **(B3)**) tells us $K \circledast (\theta \wedge \phi) \subseteq (K \circledast (\theta \wedge \phi \wedge \psi)) + \neg(\theta \wedge \phi)$. Hence using this with the assumption $\theta \in K \circledast (\theta \wedge \phi)$ yields $\theta \in (K \circledast (\phi \wedge \theta \wedge \psi)) + \neg(\theta \wedge \phi)$. By classical logic this is equivalent to the desired $\theta \in K \circledast (\phi \wedge \theta \wedge \psi)$.

Next we want to prove:

Theorem 6 Let K be a belief set and $\circledast$ an operator for K. Then $\circledast$ is a basic removal operator for K iff $\circledast$ satisfies **(B1)–(B8)**.

First let's prove the postulates are sound for basic removal.

PROOF. [Soundness] We check each postulate in turn:
(B1) $K \circledast_{(\leq,\preceq)} \phi = Cn(K \circledast_{(\leq,\preceq)} \phi)$. Obvious.
(B2) $\phi \notin K \circledast_{(\leq,\preceq)} \phi$. Since $\min([\neg\phi], \leq) \subseteq [K \circledast_{(\leq,\preceq)} \phi)]$ by Lemma A, we know there is at least *one* world in $[K \circledast_{(\leq,\preceq)} \phi)]$ satisfying $\neg\phi$. This is enough to show $\phi \notin K \circledast_{(\leq,\preceq)} \phi$.
(B3) If $\phi_1 \equiv \phi_2$ then $K \circledast_{(\leq,\preceq)} \phi_1 = K \circledast_{(\leq,\preceq)} \phi_2$. Obvious.

(B4) $K \circledast_{(\leq,\preceq)} \bot = K$. Firstly, since $\leq$ is anchored on $[K]$ we know $[K] = \min(\mathcal{W}, \leq) = \min([\neg\bot], \leq)$. By Lemma A, then, $[K] \subseteq [K \circledast_{(\leq,\preceq)} \bot]$. Meanwhile, for any $w \in \mathcal{W}$, if $w \preceq w'$ for some $w' \in [K]$ then, since $\preceq$ is a sub-relation of $\leq$, also $w \leq w'$ for some $w' \in [K]$ and so, since $\leq$ is transitive, $w \leq w''$ for *all* $w'' \in \mathcal{W}$, i.e., $w \in [K]$. Hence $[K \circledast_{(\leq,\preceq)} \bot] \subseteq [K]$ and so we have equality. This means $K \circledast_{(\leq,\preceq)} \bot = K$ as required.

(B5) If $\theta \in K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)$ then $\theta \in K \circledast_{(\leq,\preceq)} (\theta \wedge \phi \wedge \psi)$. Suppose $\theta \in K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)$. Choose any $w_0 \in \min([\neg(\theta \wedge \phi)], \leq)$. Then we first claim $w \in [\theta]$ for all $w \leq w_0$. This holds since if $w \leq w_0$ and $w \notin [\theta]$, i.e., $w \in [\neg\theta]$, then also $w \in [\neg(\theta \wedge \phi)]$ and so, using the minimality of w_0, we must have $w \in \min([\neg(\theta \wedge \phi)], \leq)$. But then since $\min([\neg(\theta \wedge \phi)], \leq) \subseteq [K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)]$ (by Lemma A) and $\theta \in K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)$ we must have $w \in [\theta]$ – contradiction. Hence $w \in [\theta]$ as claimed. Now suppose $w' \in [K \circledast_{(\leq,\preceq)} (\theta \wedge \phi \wedge \psi)]$. We must show $w' \in [\theta]$. But $w' \in [K \circledast_{(\leq,\preceq)} (\theta \wedge \phi \wedge \psi)]$ gives $w' \preceq w$ for some $w \in \min([\neg(\theta \wedge \phi \wedge \psi)], \leq)$ and so, since $\preceq$ is a sub-relation of $\leq$, $w' \leq w$ for some $w \in \min([\neg(\theta \wedge \phi \wedge \psi)], \leq)$. From $w_0 \in [\neg(\theta \wedge \phi)]$ we know $w_0 \in [\neg(\theta \wedge \phi \wedge \psi)]$ and so, by the minimality of w, $w \leq w_0$. Hence, since $\leq$ is transitive, $w' \leq w_0$. We conclude from the above claim that $w' \in [\theta]$ as required.

(B6) If $\theta \in K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)$ then $K \circledast_{(\leq,\preceq)} \phi \subseteq K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)$. Suppose $\theta \in K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)$. Then using this with Lemma A gives us $\min([\neg(\theta \wedge \phi)], \leq) \subseteq [\theta]$. This means it must be the case that $\min([\neg(\theta \wedge \phi)], \leq) \subseteq \min([\neg\phi], \leq)$, since if $w \in \min([\neg(\theta \wedge \phi)], \leq)$ then $w \in [\theta]$ so we must have $w \in [\neg\phi]$, and, since $w \leq w'$ for all $w' \in [\neg(\theta \wedge \phi)]$, we necessarily have $w \leq w'$ for all $w' \in [\neg\phi]$. So, for any w', if $w' \preceq w''$ for some $w'' \in \min([\neg(\theta \wedge \phi)], \leq)$ then we immediately get also $w' \preceq w''$ for some $w'' \in \min([\neg\phi], \leq)$. This is enough to prove $[K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)] \subseteq [K \circledast_{(\leq,\preceq)} \phi]$, which gives the required conclusion.

(B7) $(K \circledast_{(\leq,\preceq)} \theta) \cap (K \circledast_{(\leq,\preceq)} \phi) \subseteq K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)$. Since $[(K \circledast_{(\leq,\preceq)} \theta) \cap (K \circledast_{(\leq,\preceq)} \phi)] = [K \circledast_{(\leq,\preceq)} \theta] \cup [K \circledast_{(\leq,\preceq)} \phi]$, it suffices to show $[K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)] \subseteq [K \circledast_{(\leq,\preceq)} \theta] \cup [K \circledast_{(\leq,\preceq)} \phi]$. But this follows from the fact that $\min([\neg(\theta \wedge \phi)], \leq) \subseteq \min([\neg\theta], \leq) \cup \min([\neg\phi], \leq)$, which is easy to show.

(B8) If $\phi \notin K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)$ then $K \circledast_{(\leq,\preceq)} (\theta \wedge \phi) \subseteq K \circledast_{(\leq,\preceq)} \phi$. Suppose $\phi \notin K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)$. Then $[K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)] \cap [\neg\phi] \neq \emptyset$, which means there exists $w_0 \in [\neg\phi]$ such that $w_0 \preceq w'$ for some $w' \in \min([\neg(\theta \wedge \phi)], \leq)$. Since $\preceq \subseteq \leq$, it follows that $w_0 \leq w'$ for some $w' \in \min([\neg(\theta \wedge \phi)], \leq)$. Since obviously $w_0 \in [\neg(\theta \wedge \phi)]$, this implies $w_0 \in \min([\neg(\theta \wedge \phi)], \leq)$. The existence of this w_0 then implies $\min([\neg\phi], \leq) \subseteq \min([\neg(\theta \wedge \phi)], \leq)$, for if $w \in \min([\neg\phi], \leq)$ then $w \leq w_0$ and so, by transitivity of $\leq$ and the fact that clearly also $w \in [\neg(\theta \wedge \phi)]$ we get $w \in \min([\neg(\theta \wedge \phi)], \leq)$. That $\min([\neg\phi], \leq) \subseteq \min([\neg(\theta \wedge \phi)], \leq)$ is enough then to prove $[K \circledast_{(\leq,\preceq)} \phi] \subseteq [K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)]$, which gives the result.

Now we show **(B1)–(B8)** are complete. The following derived rules will be useful.

Lemma B The following four properties follow from **(B1)–(B8)**:

(X1) $K \circledast \phi \subseteq K + \neg\phi$
(X2) If $\theta \wedge \phi \in K \circledast (\theta \wedge \phi \wedge \psi)$ then $\theta \in K \circledast (\theta \wedge \psi)$
(X3) If $\theta \notin K \circledast (\theta \wedge \phi)$ and $\theta \notin K \circledast (\theta \wedge \psi)$ then $\theta \notin K \circledast (\theta \wedge \phi \wedge \psi)$
(X4) If $\theta \in K \circledast (\theta \wedge \phi)$ then $K \circledast (\theta \wedge \phi) = K \circledast \phi$.

PROOF. For **(X1)** recall that **(B5′)** is equivalent to **(B5)** by Proposition 5. Then from **(B5′)** we know $K \circledast \phi \subseteq (K \circledast (\phi \wedge \bot)) + \neg\phi$. **(X1)** then follows from applying to this **(B3)** (using $\phi \wedge \bot \equiv \bot$) followed by **(B4)** (i.e., $K \circledast \bot = K$).

For **(X2)** suppose $\theta \wedge \phi \in K \circledast (\theta \wedge \phi \wedge \psi)$. By **(B1)** and **(B2)** this means $\theta \wedge \psi \notin K \circledast (\theta \wedge \phi \wedge \psi)$. Using this with **(B8)** we get $K \circledast (\theta \wedge \phi \wedge \psi) \subseteq K \circledast (\theta \wedge \psi)$. Hence, since $\theta \in K \circledast (\theta \wedge \phi \wedge \psi)$ (which follows from the assumption $\theta \wedge \phi \in K \circledast (\theta \wedge \phi \wedge \psi)$ and **(B1)**), we get $\theta \in K \circledast (\theta \wedge \psi)$ as required.

For **(X3)** suppose $\theta \notin K \circledast (\theta \wedge \phi)$ and $\theta \notin K \circledast (\theta \wedge \psi)$. From the latter we get $\theta \wedge \phi \notin K \circledast (\theta \wedge \phi \wedge \psi)$ using **(X2)** above. Hence, by **(B8)**, $K \circledast (\theta \wedge \phi \wedge \psi) \subseteq K \circledast (\theta \wedge \phi)$ and so from $\theta \notin K \circledast (\theta \wedge \phi)$ we get $\theta \notin K \circledast (\theta \wedge \phi \wedge \psi)$ as required.

(X4) is a straightforward consequence from mainly **(B6)** and **(B8)**.

Now let's give the completeness proof of Theorem 6.

PROOF. [Completeness] Let K and $\circledast$ be given. We need to find some K-context $(\leq, \preceq)$ such that $\circledast = \circledast_{(\leq,\preceq)}$. We use the one from Definition 3 of the paper, which we denoted by $\mathcal{C}(K, \circledast)$. Recall we define the two relations $\leq, \preceq$ on $\mathcal{W}$ from K and $\circledast$ as follows:

$(\leq)$ $\quad w_1 \leq w_2$ iff $\neg w_1 \notin K \circledast (\neg w_1 \wedge \neg w_2)$

$(\preceq)$ $\quad w_1 \preceq w_2$ iff $\neg w_1 \notin K \circledast \neg w_2$

Recall that in $\neg w_1$ etc, world w_1 stands for any sentence which has w_1 as its only model. Note that, here and in what follows, the precise choice of *which* sentence is irrelevant thanks to **(B1)** and **(B3)**.

We now need to show several things: (1) $\leq$ is a total pre-order on $\mathcal{W}$, anchored on $[K]$, (2) $\preceq$ is a reflexive sub-relation of $\leq$, (3) $K \circledast \phi = K \circledast_{(\leq,\preceq)} \phi$ for all ϕ.

<u>(1) $\leq$ is a total pre-order on $\mathcal{W}$, anchored on $[K]$</u> To show $\leq$ is a total pre-order we need to show $\leq$ is transitive and complete. First we show transitivity. So suppose $w_1 \leq w_2$ and $w_2 \leq w_3$, i.e., $\neg w_1 \notin K \circledast (\neg w_1 \wedge \neg w_2)$ and $\neg w_2 \notin K \circledast (\neg w_2 \wedge \neg w_3)$. We need to show $w_1 \leq w_3$, i.e., $\neg w_1 \notin K \circledast (\neg w_1 \wedge \neg w_3)$. Equivalently we can show that if $\neg w_1 \in K \circledast (\neg w_1 \wedge \neg w_3)$ and $\neg w_2 \notin K \circledast (\neg w_2 \wedge \neg w_3)$

then $\neg w_1 \in K \circledast (\neg w_1 \wedge \neg w_2)$. But from $\neg w_2 \notin K \circledast (\neg w_2 \wedge \neg w_3)$ we get $\neg w_1 \wedge \neg w_2 \notin K \circledast (\neg w_1 \wedge \neg w_2 \wedge \neg w_3)$ by rule **(X2)** in Lemma B. Hence, by **(B8)**, $K \circledast (\neg w_1 \wedge \neg w_2 \wedge \neg w_3) \subseteq K \circledast (\neg w_1 \wedge \neg w_2)$. Meanwhile from $\neg w_1 \in K \circledast (\neg w_1 \wedge \neg w_3)$ we deduce $\neg w_1 \in K \circledast (\neg w_1 \wedge \neg w_2 \wedge \neg w_3)$ from **(B5)**. Using this with $K \circledast (\neg w_1 \wedge \neg w_2 \wedge \neg w_3) \subseteq K \circledast (\neg w_1 \wedge \neg w_2)$ gives the required $\neg w_1 \in K \circledast (\neg w_1 \wedge \neg w_2)$.

To show $\leq$ is complete we need to show either $w_1 \leq w_2$ or $w_2 \leq w_1$, i.e., either $\neg w_1 \notin K \circledast (\neg w_1 \wedge \neg w_2)$ or $\neg w_2 \notin K \circledast (\neg w_2 \wedge \neg w_1)$. This follows easily from **(B1)** and **(B2)**.

It remains to show $\leq$ is anchored on $[K]$, i.e., $\min(\mathcal{W}, \leq) = [K]$. Let $w \in \min(\mathcal{W}, \leq)$. Then $\neg w \notin K \circledast (\neg w \wedge \neg w')$ for all $w' \in \mathcal{W}$. By repeated use of rule **(X3)** from Lemma B we obtain from this $\neg w \notin K \circledast \bigwedge_{w' \in \mathcal{W}} \neg w'$. Since $\bot \equiv \bigwedge_{w' \in \mathcal{W}} \neg w'$ this gives $\neg w \notin K \circledast \bot$ by **(B3)**. Hence, since $K \circledast \bot = K$ by **(B4)**, this gives in turn $\neg w \notin K$, which is equivalent to $w \in [K]$. Thus we have shown $\min(\mathcal{W}, \leq) \subseteq [K]$. For the converse direction suppose $w \notin \min(\mathcal{W}, \leq)$. Then $\neg w \in K \circledast (\neg w \wedge \neg w')$ for some $w' \in \mathcal{W}$. Using this with **(X1)** gives $\neg w \in K + (w \vee w')$, equivalently $(w \vee w') \rightarrow \neg w \in K$. Since $\neg w \equiv ((w \vee w') \rightarrow \neg w)$, this is equivalent to $\neg w \in K$, i.e., $w \notin [K]$ as required.

<u>(2) $\preceq$ is a reflexive sub-relation of $\leq$</u> First, to show $\preceq$ is reflexive we need to show $\neg w \notin K \circledast \neg w$ for all $w \in \mathcal{W}$. This is immediate from **(B2)**. To show $\preceq$ is a sub-relation of $\leq$ we need to show $w_1 \preceq w_2$ implies $w_1 \leq w_2$. We show the contrapositive. So suppose $w_1 \not\leq w_2$. Since we have already shown above that $\leq$ is connected, this means we must have $w_2 \leq w_1$, i.e., $\neg w_2 \notin K \circledast (\neg w_1 \wedge \neg w_2)$. Hence, by **(B8)**, $K \circledast (\neg w_1 \wedge \neg w_2) \subseteq K \circledast \neg w_2$. Our assumption $w_1 \not\leq w_2$ yields $\neg w_1 \in K \circledast (\neg w_1 \wedge \neg w_2)$, and so using this with $K \circledast (\neg w_1 \wedge \neg w_2) \subseteq K \circledast \neg w_2$ gives $\neg w_1 \in K \circledast \neg w_2$, i.e., $w_1 \not\preceq w_2$ as required.

<u>(3) $K \circledast \phi = K \circledast_{(\leq,\preceq)} \phi$ for all ϕ</u> Let $\phi \in L$. We will show $[K \circledast \phi] = \{w \mid w \preceq w' \text{ for some } w' \in \min([\neg\phi], \leq)\}$. Let $[\neg\phi] = \{x_1, \ldots, x_m\}$ and let $S = \{i \mid x_i \in \min([\neg\phi], \leq)\}$. We will first show

$$K \circledast \phi = K \circledast \bigwedge_{i \in S} \neg x_i.$$

To see this first note that $K \circledast \phi = K \circledast \bigwedge_{i=1}^{m} \neg x_i = K \circledast (\bigwedge_{i \in S} \neg x_i) \wedge (\bigwedge_{j \notin S} \neg x_j)$ by **(B3)**. Now let $j \notin S$. Then we know there exists $i \in S$ such that $x_i < x_j$ (otherwise $j \in S$), so $\neg x_j \in K \circledast (\neg x_i \wedge \neg x_j)$ for some $i \in S$. Using this with **(B5)** gives $\neg x_j \in K \circledast (\bigwedge_{i \in S} \neg x_i) \wedge (\bigwedge_{i \notin S} \neg x_i)$. Since this holds for each $j \notin S$ we obtain $\bigwedge_{j \notin S} \neg x_j \in K \circledast (\bigwedge_{i \in S} \neg x_i) \wedge (\bigwedge_{j \notin S} \neg x_j)$ by **(B1)**. Hence, from rule **(X4)** in Lemma B, we get $K \circledast (\bigwedge_{i \in S} \neg x_i) \wedge (\bigwedge_{j \notin S} \neg x_j) = K \circledast \bigwedge_{i \in S} \neg x_i$, i.e., $K \circledast \phi = K \circledast \bigwedge_{i \in S} \neg x_i$ as required.

Now suppose $w \in [K \circledast \phi]$. We must show there exists $i \in S$ such that $w \preceq x_i$. But from $w \in [K \circledast \phi]$ we get $\neg w \notin K \circledast \phi$. From the above, this is the same as $\neg w \notin K \circledast \bigwedge_{i \in S} \neg x_i$. By **(B7)** this means we must have $\neg w \notin K \circledast \neg x_i$ for some $i \in S$, i.e., $w \preceq x_i$ as required.

For the converse direction, choose $i \in S$ such that $w \preceq x_i$. Then $\neg w \notin K \circledast \neg x_i$. Now, for all $i' \in S$ such that $i' \neq i$ we have $x_i \leq x_{i'}$, i.e., $\neg x_i \notin K \circledast (\neg x_i \wedge \neg x_{i'})$. By repeated application of rule **(X3)** from Lemma B we obtain $\neg x_i \notin K \circledast \bigwedge_{i' \in S} \neg x_{i'}$. Hence, from **(B8)**, $K \circledast \bigwedge_{i' \in S} \neg x_{i'} \subseteq K \circledast \neg x_i$. Since $\neg w \notin K \circledast \neg x_i$, this gives us $\neg w \notin K \circledast \bigwedge_{i' \in S} \neg x_{i'} = K \circledast \phi$, and so $w \in [K \circledast \phi]$ as required.

B Proofs for Section 3

Proposition 9 *(i).* If $(\leq, \preceq)$ is transitive then $\circledast_{(\leq,\preceq)}$ satisfies:

(BTran) If $K \circledast \theta \subseteq (K \circledast \phi) + \neg\phi$ then $K \circledast \theta \subseteq K \circledast \phi$

(ii). If $\circledast$ is a removal operator satisfying **(BTran)** then the relation $\preceq$ of $\mathcal{C}(K, \circledast)$ is transitive.

PROOF. *(i).* Suppose $K \circledast_{(\leq,\preceq)} \theta \subseteq (K \circledast_{(\leq,\preceq)} \phi) + \neg\phi$, equivalently $[K \circledast_{(\leq,\preceq)} \phi] \cap [\neg\phi] \subseteq [K \circledast_{(\leq,\preceq)} \theta]$, and let $w \in [K \circledast_{(\leq,\preceq)} \phi]$. We must show $w \in [K \circledast_{(\leq,\preceq)} \theta]$. But from $w \in [K \circledast_{(\leq,\preceq)} \phi]$ we know $w \preceq w'$ for some $w' \in \min([\neg\phi], \leq)$. By reflexivity of $\preceq$ we know $w' \in [K \circledast_{(\leq,\preceq)} \phi]$ and obviously also $w' \in [\neg\phi]$. From the assumption $[K \circledast_{(\leq,\preceq)} \phi] \cap [\neg\phi] \subseteq [K \circledast_{(\leq,\preceq)} \theta]$ we deduce $w' \in [K \circledast_{(\leq,\preceq)} \theta]$, i.e., $w' \preceq w''$ for some $w'' \in \min([\neg\theta], \leq)$. Then by transitivity of $\preceq$ we get also $w \preceq w''$ and so $w \in [K \circledast_{(\leq,\preceq)} \theta]$ as required.

(ii). Suppose $\neg w \notin K \circledast \neg w'$ and $\neg w' \notin K \circledast \neg w''$. Using **(B1)** this is equivalent to $w \in [K \circledast \neg w']$ and $w' \in [K \circledast \neg w'']$. We must show $w \in [K \circledast \neg w'']$. But from $w' \in [K \circledast \neg w'']$ and the fact $[w'] = \{w\}$ we know $[K \circledast \neg w'] \cap [w'] \subseteq [K \circledast \neg w'']$. Applying **(BTran)** to this yields $[K \circledast \neg w'] \subseteq [K \circledast \neg w'']$ (with a little help from **(B1)**) from which we obtain the required $w \in [K \circledast \neg w'']$ from $w \in [K \circledast \neg w']$.

Proposition 10 Any removal operator which satisfies **(BTran)** and **(B5′)** also satisfies **(B6)**.

PROOF. Suppose $\theta \in K \circledast (\theta \wedge \phi)$. To show $K \circledast \phi \subseteq K \circledast (\theta \wedge \phi)$ it suffices by **(BTran)** to show $K \circledast \phi \subseteq (K \circledast (\theta \wedge \phi)) + \neg(\theta \wedge \phi)$. But from the assumption $\theta \in K \circledast (\theta \wedge \phi)$ we know $(K \circledast (\theta \wedge \phi)) + \neg(\theta \wedge \phi) = (K \circledast (\theta \wedge \phi)) + \neg\phi$, which contains $K \circledast \phi$ by **(B5′)** as required.

Recall the condition (a) on $(\leq, \preceq)$:

(a) If $w_1 \sim w_2$ and $w_1 \preceq w_2$ then $w_2 \preceq w_1$

Proposition 11 *(i).* If $(\leq, \preceq)$ satisfies (a) then $\divideontimes_{(\leq,\preceq)}$ satisfies:

(BPriority) If $\theta \in K \divideontimes \phi$ and $\phi \notin K \divideontimes \theta$ then $\theta \in K \divideontimes (\theta \wedge \phi)$

(ii). If $\divideontimes$ is a removal operator satisfying **(BPriority)** then $\mathcal{C}(K, \divideontimes)$ satisfies (a).

PROOF. *(i).* Suppose $\theta \in K \divideontimes_{(\leq,\preceq)} \phi$, $\phi \notin K \divideontimes_{(\leq,\preceq)} \theta$ and, for contradiction, $\theta \notin K \divideontimes_{(\leq,\preceq)} (\phi \wedge \theta)$. Then from $\phi \notin K \divideontimes_{(\leq,\preceq)} \theta$ we know from **(B1)** that $[K \divideontimes_{(\leq,\preceq)} \theta] \not\subseteq [\phi]$ and so there exist w_1, w_2 such that $w_1 \in [\neg\phi]$, $w_1 \preceq w_2$ and $w_2 \in \min([\neg\theta], \leq)$, while from $\theta \notin K \divideontimes_{(\leq,\preceq)} (\phi \wedge \theta)$ we know there exist z, w such that $z \in [\neg\theta]$, $z \preceq w$ and $w \in \min([\neg(\phi \wedge \theta)], \leq)$. We have the following inequalities:

$$w \leq w_1 \preceq w_2 \leq z \preceq w.$$

To see this, note $w_1 \preceq w_2$ and $z \preceq w$ are both given, $w \leq w_1$ follows since $w_1 \in [\neg\phi]$ and the minimality of w, while $w_2 \leq z$ follows from $z \in [\neg\theta]$ and the minimality of w_2. Since $\preceq \subseteq \leq$ this yields $w \leq w_1 \leq w_2 \leq z \leq w$, thus $w \sim w_1 \sim w_2 \sim z$, and in particular $w_1 \sim w_2$. Using this with $w_1 \preceq w_2$ and property (a) yields $w_2 \preceq w_1$. Now, the above proved inequality also gives $w_1 \leq w$, which using the minimality of w is enough to show $w_1 \in \min([\neg\phi], \leq)$. Hence $w_2 \in [K \divideontimes_{(\leq,\preceq)} \phi]$. But $w_2 \in [\neg\theta]$, contradicting $\theta \in K \divideontimes_{(\leq,\preceq)} \phi$.

(ii). We will show $w_1 \preceq w_2$ and $w_2 \not\preceq w_1$ implies $w_1 < w_2$. So suppose $w_1 \preceq w_2$ and $w_2 \not\preceq w_1$, i.e., by definition of $\mathcal{C}(K, \divideontimes)$, $\neg w_1 \notin K \divideontimes \neg w_2$ and $\neg w_2 \in K \divideontimes \neg w_1$. Then applying **(BPriority)** (and **(B3)**) to this gives $\neg w_2 \in K \divideontimes (\neg w_1 \wedge \neg w_2)$,or i.e., $w_1 < w_2$ as required.

Proposition 12 *(i).* If $(\leq, \preceq)$ is both transitive and satisfies (a) then $\divideontimes_{(\leq,\preceq)}$ satisfies:

(BConserv) If $K \divideontimes \theta \not\subseteq K \divideontimes \phi$ then there exists $\lambda \in L_*$ such that

$\phi \vdash \lambda$ and $(K \divideontimes \theta) \cup (K \divideontimes \lambda) \vdash \phi$

(ii). If $\divideontimes$ is a removal operator satisfying **(BConserv)** then $\mathcal{C}(K, \divideontimes)$ is transitive and satisfies (a).

PROOF. *(i).* Let $(\leq, \preceq)$ be a transitive context which satisfies (a). Suppose $K \divideontimes_{(\leq,\preceq)} \theta \not\subseteq K \divideontimes_{(\leq,\preceq)} \phi$. Then since $\divideontimes_{(\leq,\preceq)}$ satisfies **(BTran)** by Proposition 9*(i)* we know $K \divideontimes_{(\leq,\preceq)} \theta \not\subseteq (K \divideontimes_{(\leq,\preceq)} \phi) + \neg\phi$. Now let $\divideontimes(\phi) \in L$, resp. $\divideontimes(\theta)$, denote some sentence such that $K \circledast_{(\leq,\preceq)} \phi = Cn(\circledast(\phi))$, resp. $K \circledast_{(\leq,\preceq)} \theta = Cn(\circledast(\theta))$. So from $K \circledast_{(\leq,\preceq)} \theta \nsubseteq (K \circledast_{(\leq,\preceq)} \phi) + \neg\phi$ we know $\circledast(\phi) \wedge \neg\phi \nvdash \circledast(\theta)$. Let $\lambda = \neg(\circledast(\phi) \wedge \neg\phi \wedge \neg \circledast (\theta))$. Then we know $\lambda \in L_*$ and $\phi \vdash \lambda$. We will show $\circledast(\lambda) \wedge \circledast(\theta) \vdash \phi$, which will suffice. Let $w_1 \in [\circledast(\lambda) \wedge \circledast(\theta)]$. We must show $w_1 \in [\phi]$. But from $w_1 \in [\circledast(\lambda)]$ we know $w_1 \preceq w_2$ for some $w_2 \in \min([\neg\lambda], \leq)$. Since $w_2 \in [\neg\lambda]$ and by definition of λ we know $w_2 \in [\circledast(\phi) \wedge \neg\phi] = [K \circledast_{(\leq,\preceq)} \phi] \cap [\neg\phi]$ and so, $w_2 \in \min([\neg\phi], \leq)$ by Lemma A. Now suppose for contradiction $w_1 \in [\neg\phi]$. Then the minimality of w_2 gives $w_2 \leq w_1$. Using this with $w_1 \preceq w_2$ and (a) yields $w_2 \preceq w_1$. Now since also $w_1 \in [\circledast(\theta)]$ we know $w_1 \preceq w_3$ for some $w_3 \in \min([\neg\theta], \leq)$. So from this and transitivity we obtain $w_2 \preceq w_3$ and thus $w_2 \in [\circledast(\theta)]$. But, looking at the definition of λ, this contradicts $w_2 \in [\neg\lambda]$. Hence $w_1 \in [\phi]$ as required.

(ii). To show $\preceq$ is transitive we need to show that if $\neg w_1 \notin K \circledast \neg w_2$ and $\neg w_2 \notin K \circledast \neg w_3$ then $\neg w_1 \notin K \circledast \neg w_3$. Equivalently, if $\neg w_1 \notin K \circledast \neg w_2$ and $\neg w_1 \in K \circledast \neg w_3$ then $\neg w_2 \in K \circledast \neg w_3$. But if $\neg w_1 \notin K \circledast \neg w_2$ and $\neg w_1 \in K \circledast \neg w_3$ then $K \circledast \neg w_3 \nsubseteq K \circledast \neg w_2$. Hence, by **(BConserv)**, there exists $\lambda \in L_*$ such that $\neg w_2 \vdash \lambda$ and $(K \circledast \neg w_3) \cup (K \circledast \lambda) \vdash \neg w_2$. Now, since λ is not a tautology, it follows from $\neg w_2 \vdash \lambda$ that in fact $\neg w_2 \equiv \lambda$ (because the only sentences strictly weaker than $\neg w_2$ are the tautologies). So, using **(B3)**, $K \circledast \lambda = K \circledast \neg w_2$. Hence we may rewrite $(K \circledast \neg w_3) \cup (K \circledast \lambda) \vdash \neg w_2$ as $(K \circledast \neg w_3) \cup (K \circledast \neg w_2) \vdash \neg w_2$, which in turn is equivalent to:

$$[K \circledast \neg w_3] \cap [K \circledast \neg w_2] \subseteq [\neg w_2]. \tag{B.1}$$

Clearly $w_2 \notin [\neg w_2]$, which from the above means w_2 cannot be an element of both $[K \circledast \neg w_3]$ *and* $[K \circledast \neg w_2]$. By **(B2)** we know $\neg w_2 \notin K \circledast \neg w_2$, i.e., $w_2 \in [K \circledast \neg w_2]$. Hence it must be that $w_2 \notin [K \circledast \neg w_3]$, i.e., $\neg w_2 \in K \circledast \neg w_3$ as required.

It remains to show condition (a) is satisfied. So suppose $w_1 \sim w_2$ and $w_1 \preceq w_2$. We must show $w_2 \preceq w_1$. In fact we will show that [$w_1 \sim w_2$ and $w_2 \npreceq w_1$] implies $w_1 \npreceq w_2$. So suppose $w_1 \sim w_2$ and $w_2 \npreceq w_1$, i.e., $\neg w_1 \notin K \circledast (\neg w_1 \wedge \neg w_2)$, $\neg w_2 \notin K \circledast (\neg w_2 \wedge \neg w_1)$ and $\neg w_2 \in K \circledast \neg w_1$. Then since $\neg w_2 \in (K \circledast \neg w_1) \setminus (K \circledast (\neg w_2 \wedge \neg w_1))$ we know $K \circledast \neg w_1 \nsubseteq K \circledast (\neg w_2 \wedge \neg w_1)$, so we may apply **(BConserv)** to deduce the existence of $\lambda \in L_*$ such that $\neg w_2 \wedge \neg w_1 \vdash \lambda$ and $(K \circledast \neg w_1) \cup (K \circledast \lambda) \vdash \neg w_2 \wedge \neg w_1$. Since obviously $w_1 \notin [\neg w_2 \wedge \neg w_1]$, this latter implies in particular that $w_1 \notin [K \circledast \neg w_1] \cap [K \circledast \lambda]$. And since we know $w_1 \in [K \circledast \neg w_1]$ by **(B2)**, we deduce from this $w_1 \notin [K \circledast \lambda]$, i.e., $\neg w_1 \in K \circledast \lambda$. Now, since $\neg w_2 \wedge \neg w_1 \vdash \lambda$ and λ is not a tautology, it must be the case that either *(i)* $\lambda \equiv \neg w_1 \wedge \neg w_2$, or *(ii)* $\lambda \equiv \neg w_1$, or *(iii)* $\lambda \equiv \neg w_2$. We show *(i)* and *(ii)* lead to contradictions, leaving *(iii)* as the only possibility, from which we then deduce $\neg w_1 \in K \circledast \neg w_2$ (using **(B3)**), i.e., $w_1 \npreceq w_2$ as required. But if *(i)* holds then $\neg w_1 \in K \circledast (\neg w_1 \wedge \neg w_2)$, contradicting our initial assumption $w_1 \sim w_2$, while if *(ii)* holds then $\neg w_1 \in K \circledast \neg w_1$, contradicting **(B2)**. This completes the proof.

Recall the postulate **(BSConserv)**:

(BSConserv) If $K \circledast \theta \not\subseteq K \circledast \phi$ then $(K \circledast \theta) \cup (K \circledast \phi) \vdash \phi$

and the condition (b) on $(\leq, \preceq)$:

(b) If $w_1 \sim w_2$ then $w_1 \preceq w_2$

Proposition 13 *(i)*. If $(\leq, \preceq)$ is transitive and satisfies (b) then $\circledast_{(\leq,\preceq)}$ satisfies **(BSConserv)**. *(ii)*. If $\circledast$ is a removal operator satisfying **(BSConserv)** then $\mathcal{C}(K, \circledast)$ is transitive and satisfies (b).

PROOF. *(i)*. Suppose $(\leq, \preceq)$ is transitive and satisfies (b), and suppose $K \circledast_{(\leq,\preceq)} \theta \not\subseteq K \circledast_{(\leq,\preceq)} \phi$. Since $\circledast_{(\leq,\preceq)}$ satisfies **(BTran)** by Proposition 9*(i)* this implies $K \circledast_{(\leq,\preceq)} \theta \not\subseteq (K \circledast_{(\leq,\preceq)} \phi) + \neg\phi$. Hence there exists w' such that $w' \in [K \circledast_{(\leq,\preceq)} \phi] \cap [\neg\phi]$ but $w' \notin [K \circledast_{(\leq,\preceq)} \theta]$. We must show $[K \circledast_{(\leq,\preceq)} \theta] \cap [K \circledast_{(\leq,\preceq)} \phi] \subseteq [\phi]$. Suppose for contradiction there exists $w \in [K \circledast_{(\leq,\preceq)} \theta] \cap [K \circledast_{(\leq,\preceq)} \phi] \cap [\neg\phi]$. Then both w and w' are elements of $[K \circledast_{(\leq,\preceq)} \phi] \cap [\neg\phi] = \min([\neg\phi], \leq)$ by Lemma A, so $w \sim w'$ which implies $w' \preceq w$ by (b). Meanwhile from $w \in [K \circledast_{(\leq,\preceq)} \theta]$ we know $w \preceq w''$ for some $w'' \in \min([\neg\theta], \leq)$. By transitivity of $\preceq$ we deduce $w' \preceq w''$ and so $w' \in [K \circledast_{(\leq,\preceq)} \theta]$, giving the required contradiction.

(ii): Suppose $\circledast$ satisfies **(BSConserv)**. Since **(BSConserv)** clearly implies **(BConserv)**, we know $(\leq, \preceq)$ is transitive by Proposition 12*(ii)*. It remains to show (b) holds. To show $w_1 \sim w_2$ implies $w_1 \preceq w_2$ we need to show that if $\neg w_1 \notin K \circledast (\neg w_1 \wedge \neg w_2)$ and $\neg w_2 \notin K \circledast (\neg w_2 \wedge \neg w_1)$ then $\neg w_1 \notin K \circledast \neg w_2$. But if $\neg w_1 \in (K \circledast \neg w_2) \setminus (K \circledast (\neg w_1 \wedge \neg w_2))$ then $K \circledast \neg w_2 \not\subseteq K \circledast (\neg w_1 \wedge \neg w_2)$, so we may apply **(BSConserv)** to deduce $(K \circledast \neg w_2) \cup (K \circledast (\neg w_1 \wedge \neg w_2)) \vdash \neg w_1 \wedge \neg w_2$. Since $w_2 \notin [\neg w_1 \wedge \neg w_2]$, this gives us $w_2 \notin [K \circledast \neg w_2] \cap [K \circledast (\neg w_1 \wedge \neg w_2)]$. But we know $w_2 \in [K \circledast \neg w_2]$ by **(B2)**, hence we must have $w_2 \notin [K \circledast (\neg w_1 \wedge \neg w_2)]$, i.e., $\neg w_2 \in K \circledast (\neg w_1 \wedge \neg w_2)$.

C Proofs for Section 4

Proposition 14 Let $\circledast$ be a basic removal operator for K, then $\circledast$ satisfies: If $\phi \notin K$ then $K \subseteq K \circledast \phi$

PROOF. First note that by **(B3)** and **(B4)**, $K = K \circledast (\bot \wedge \phi)$. The rule then follows as an instance of **(B8)** (substitute $\bot$ for θ there).

Proposition 15 *(i)*. If $(\leq, \preceq)$ satisfies (c) then $\circledast_{(\leq,\preceq)}$ satisfies **(Vacuity)**. *(ii)*. If $\circledast$

is a removal operator satisfying **(Vacuity)** then $\mathcal{C}(K, \circledast)$ satisfies (c).

PROOF. *(i)*: Suppose $(\leq, \preceq)$ satisfies (c). We only need to show the 'missing half' of **(Vacuity)**: If $\phi \notin K$ then $K \circledast_{(\leq,\preceq)} \phi \subseteq K$. So suppose $\phi \notin K$. Then there exists $w_0 \in [K] \cap [\neg\phi]$. Since $\leq$ is anchored on $[K]$, $w_0 \in \min([\neg\phi], \leq)$. Since $w_0 \in [K]$, we know by (c) that $w \preceq w_0$ for all $w \in [K]$. Thus every world in $[K]$ is $\preceq$-below some element of $\min([\neg\phi], \leq)$ (namely w_0). Hence $[K] \subseteq [K \circledast_{(\leq,\preceq)} \phi]$ which gives the required conclusion.

(ii): Suppose $\circledast$ satisfies **(Vacuity)**, and suppose $w_1, w_2 \in [K]$. $w_2 \in [K]$ gives $\neg w_2 \notin K$. Hence, from **(Vacuity)**, $K \circledast \neg w_2 \subseteq K$, i.e., $[K] \subseteq [K \circledast \neg w_2]$. Using this with $w_1 \in [K]$ yields $w_1 \in [K \circledast \neg w_2]$, which entails the required $\neg w_1 \notin K \circledast \neg w_2$.

Recall the definition of $\div'$ from $\circledast$:

$$K \div' \phi = \begin{cases} K & \text{if } \phi \notin K \\ K \circledast \phi & \text{otherwise.} \end{cases}$$

Proposition 16 If $(\leq, \preceq)$ is the K-context corresponding to $\circledast$, then the K-context corresponding to $\div'$ defined above is $(\leq, \preceq')$, where $\preceq'$ is obtained from $\preceq$ by setting $w_1 \preceq' w_2$ iff $w_1 \preceq w_2$ or $w_1, w_2 \in [K]$.

PROOF. First consider the case $\phi \notin K$. In this case by Proposition 14 we already know $K \subseteq K \circledast_{(\leq,\preceq')} \phi$, so we just need to show $K \circledast_{(\leq,\preceq')} \phi \subseteq K$, i.e., $[K] \subseteq [K \circledast_{(\leq,\preceq')} \phi]$. So let $w \in [K]$. Since $\phi \notin K$ there exists $w' \in [K] \cap [\neg\phi]$. Clearly since $\leq$ is anchored on $[K]$ we must have $w' \in \min([\neg\phi], \leq)$, while since $w, w' \in [K]$ we have $w \preceq' w'$ from the definition of $\preceq'$. Hence $w \in K \circledast_{(\leq,\preceq')} \phi$ as required.

Now consider the case $\phi \in K$. In this case we must show $[K \circledast_{(\leq,\preceq')} \phi] = [K \circledast_{(\leq,\preceq)} \phi]$. The right-to-left inclusion is immediate from the fact $\preceq \subseteq \preceq'$ by definition of $\preceq'$. For the left-to-right inclusion suppose $w \in [K \circledast_{(\leq,\preceq')} \phi]$. Then $w \preceq' w'$ for some $w' \in \min([\neg\phi], \leq)$. Since $w' \in [\neg\phi]$ and $\phi \in K$ we know $w' \notin [K]$, so by definition of $\preceq'$ we must have $w \preceq w'$. Hence $w \in [K \circledast_{(\leq,\preceq)} \phi]$ as required.

Recall the definition of $\preceq''$ from $\preceq$: $w_1 \preceq'' w_2$ iff either $w_1 \preceq w_2$ or $[w_1 \in [K]$ and $w' \preceq w_2$ for *some* $w' \in [K]]$, and recall that if $\circledast$ is the operator corresponding to $(\leq, \preceq)$, then $\mathbb{V}(\circledast)$ is the operator corresponding to $(\leq, \preceq'')$.

Proposition 17 Let $\circledast$ be a removal operator for K corresponding to K-context $(\leq, \preceq)$ and let $\div = \mathbb{V}(\circledast)$. Then

$$K \div \phi = \begin{cases} K \cap K \circledast \phi & \text{if } K \cup K \circledast \phi \text{ is consistent} \\ K \circledast \phi & \text{otherwise} \end{cases}$$

PROOF. First consider the case $K \cup K \circledast \phi$ is consistent, i.e., $[K] \cap [K \circledast \phi] \neq \emptyset$. We must show $K \div \phi = K \cap K \circledast \phi$, equivalently $[K \div \phi] = [K] \cup [K \circledast \phi]$. For the left-to-right inclusion suppose $w \in [K \div \phi]$ and $w \notin [K]$. Then $w \preceq'' w'$ for some $w' \in \min([\neg\phi], \leq)$. Since $w \notin [K]$ the definition of $\preceq''$ gives $w \preceq w'$ and so $w \in [K \circledast \phi]$ as required. For the right-to-left inclusion, the fact $[K \div \phi] \supseteq [K \circledast \phi]$ is immediate from the fact $\preceq \subseteq \preceq''$ by definition of $\preceq''$. It remains to prove $[K \div \phi] \supseteq [K]$. So suppose $w \in [K]$. Since we assume $[K] \cap [K \circledast \phi] \neq \emptyset$ there exists $w_0 \in [K] \cap [K \circledast \phi]$, i.e., $w_0 \in [K]$ and $w_0 \preceq w'$ for some $w' \in \min([\neg\phi], \leq)$. By definition of $\preceq''$ all this gives $w \preceq'' w'$ and so $w \in [K \div \phi]$ as required.

Now consider the case $[K] \cap [K \circledast \phi] = \emptyset$. We must show $[K \div \phi] = [K \circledast \phi]$. Once more the right-to-left inclusion is immediate from $\preceq \subseteq \preceq''$. For the converse direction suppose $w \in [K \div \phi]$. Then $w \preceq'' w'$ for some $w' \in \min([\neg\phi], \leq)$. By definition of $\preceq''$ we know either $w \preceq w'$ or [$w \in [K]$ and $w'' \preceq w'$ for some $w'' \in [K]$]. But this latter case would give $w'' \in [K] \cap [K \circledast \phi]$, contrary to our assumption $[K] \cap [K \circledast \phi] = \emptyset$. Hence we must be in the former case $w \preceq w'$, which implies $w \in [K \circledast \phi]$ as required.

Proposition 18 *(i).* If $(\leq, \preceq)$ satisfies (d) then $\circledast_{(\leq,\preceq)}$ satisfies **(Inclusion)**. *(ii).* If $\circledast$ is a removal operator satisfying **(Inclusion)** then $\mathcal{C}(K, \circledast)$ satisfies (d).

PROOF. *(i)*: Suppose $(\leq, \preceq)$ satisfies (d). Then since every element of $[K]$ is $\preceq$-below every world in $\mathcal{W}$, it is clear that $[K] \subseteq [K \circledast_{(\leq,\preceq)} \phi]$ for all ϕ, i.e., $K \circledast_{(\leq,\preceq)} \phi \subseteq K$ as required.

(ii): Suppose $\circledast$ satisfies **(Inclusion)** and suppose $w_1 \in [K]$. Let $w_2 \in \mathcal{W}$. Then $[K] \subseteq [K \circledast \neg w_2]$ by **(Inclusion)** so $w_1 \in [K \circledast \neg w_2]$, i.e., $\neg w_1 \notin K \circledast \neg w_2$. Thus $w_1 \preceq w_2$ as required.

Recall that the incarceration $\dot{-}$ of a removal operator $\circledast$ is defined by setting, for each $\phi \in L_*$,

$$K \dot{-} \phi = K \cap K \circledast \phi.$$

Proposition 19 If $(\leq, \preceq)$ is the K-context corresponding to $\divideontimes$, then the K-context corresponding to $\dot{-}$ is $(\leq, \preceq'')$, where $\preceq''$ is obtained from $\preceq$ by setting $w_1 \preceq'' w_2$ iff $w_1 \preceq w_2$ or $w_1 \in [K]$. Furthermore:
(i). If $\preceq$ is transitive then so is $\preceq''$.
(ii). If $\preceq$ satisfies (a) then so does $\preceq''$.
(iii). If $\preceq$ satisfies (b) then so does $\preceq''$.

PROOF. First we show $[K \divideontimes_{(\leq,\preceq'')} \phi] = [K] \cup [K \divideontimes_{(\leq,\preceq)} \phi]$. We have that w is an element of the left-hand-side iff $w \preceq'' w'$ for some $w' \in \min([\neg\phi], \leq)$. But from the definition of $\preceq''$ this latter is the same as saying that either $w \in [K]$ or $w \preceq w'$ for some $w' \in \min([\neg\phi], \leq)$, i.e., $w \in [K] \cup [K \divideontimes_{(\leq,\preceq)} \phi]$ as required.

(i). Suppose $\preceq$ is transitive and that $w_1 \preceq'' w_2$ and $w_2 \preceq'' w_3$. We must show $w_1 \preceq'' w_3$. If $w_1 \in [K]$ then we get the required conclusion, so suppose $w_1 \notin [K]$. Then from $w_1 \preceq'' w_2$ we know $w_1 \preceq w_2$. Since $w_1 \notin [K]$ we must also have $w_2 \notin [K]$ (because $\preceq \subseteq \leq$ so from $w_1 \preceq w_2$ we know $w_1 \leq w_2$), and so from $w_2 \preceq'' w_3$ we get $w_2 \preceq w_3$. Hence we obtain the desired $w_1 \preceq w_3$ by applying the transitivity of $\preceq$.

(ii). Suppose $\preceq$ satisfies (a) and suppose $w_1 \sim w_2$ and $w_1 \preceq'' w_2$. We must show $w_2 \preceq'' w_1$. Note that, since $w_1 \sim w_2$, we have $w_1 \in [K]$ iff $w_2 \in [K]$. If $w_2 \in [K]$ then we obtain $w_2 \preceq'' w_1$ immediately, while if $w_2 \notin [K]$ then also $w_1 \notin [K]$, and then the desired conclusion follows from the assumption that $\preceq$ satisfies (a).

(iii). Follows from similar reasoning to part *(ii)* above.

Proposition 20 *(i).* If $(\leq, \preceq)$ satisfies (e) then $\divideontimes_{(\leq,\preceq)}$ satisfies **(Recovery)**. *(ii).* If $\divideontimes$ is a removal operator satisfying **(Recovery)** then $\mathcal{C}(K, \divideontimes)$ satisfies (e).

PROOF. For this proof, first note the **(Recovery)** rule is equivalent to $[K \divideontimes \phi] \cap [\phi] \subseteq [K]$.

(i): Suppose $(\leq, \preceq)$ satisfies (e). Let $w \in [K \divideontimes \phi] \cap [\phi]$. Then, since $w \in [K \divideontimes \phi]$, $w \preceq w'$ for some $w' \in \min([\neg\phi], \leq)$. Since $w \in [\phi]$ and $w' \in [\neg\phi]$ we cannot have $w = w'$. Hence, by (e), $w \in [K]$ as required.

(ii): Suppose $\divideontimes$ satisfies **(Recovery)** and suppose $w_1 \preceq w_2$. Then $w_1 \in [K \divideontimes \neg w_2]$. We need to show either $w_1 = w_2$ or $w_1 \in [K]$. But if $w_1 \neq w_2$ then $w_1 \in [\neg w_2]$, and so we can conclude $w_1 \in [K]$ using **(Recovery)**.

Proposition 21 The following are equivalent:
(i). $\divideontimes$ is a full AGM contraction operator (i.e., satisfying the basic and supplemen-

tary AGM postulates).
(ii). $\divideontimes$ satisfies **(B1)–(B8)** plus **(Inclusion)** and **(Recovery)**.
(iii). $\divideontimes = \divideontimes_{(\leq,\preceq)}$ for some $(\leq, \preceq)$ which satisfies (d) and (e).

PROOF. Recall that (d) + (e) specifies $\preceq$ uniquely in terms of $\leq$. Then equivalence *(i)*⇔*(ii)* follows from the well-established representation results relating full AGM contraction to total pre-orders over worlds [4,5]. The equivalence *(ii)*⇔*(iii)* follows from Theorem 6 and Propositions 18 and 20.

D Proofs for Section 5

Proposition 22 $\divideontimes$ is a linear liberation operator iff it is a basic removal operator which satisfies **(Hyperreg)**.

PROOF. We need to show the list of postulates for linear liberation is equivalent to **(B1)–(B8)** plus **(Hyperreg)**. So first suppose **(B1)–(B3)** hold together with **(Vacuity)** and **(Hyperreg)**. Then **(B4)** holds since it is implied by **(Vacuity)**. To see **(B5)** holds we show the contrapositive. So suppose $\theta \notin K \divideontimes (\theta \wedge \phi \wedge \psi)$. Then, by **(B1)**, $\theta \wedge \phi \notin K \divideontimes (\theta \wedge \phi \wedge \psi)$. Hence $K \divideontimes (\theta \wedge \phi \wedge \psi) = K \divideontimes (\theta \wedge \phi)$ by **(Hyperreg)**, and so the desired $\theta \notin K \divideontimes (\theta \wedge \phi)$ follows. **(B6)** and **(B8)** follow straightforwardly using **(Hyperreg)**, while **(B7)** also holds easily, once it is noticed that the Decomposition property holds for all linear liberation operators either $K \divideontimes (\theta \wedge \phi)$ equals either $K \divideontimes \theta$ or $K \divideontimes \phi$.

For the other direction it amounts to showing that the addition of **(Hyperreg)** to the basic removal postulates allows the derivation of **(Vacuity)**. This follows by noticing $K = K \divideontimes (\bot \wedge \phi)$ by **(B3)** and **(B4)**. **(Vacuity)** is then seen to be just an instance of **(Hyperreg)**.

Proposition 23 *(i).* If $(\leq, \preceq)$ satisfies (f) then $\divideontimes_{(\leq,\preceq)}$ satisfies **(Hyperreg)**. *(ii).* If $\divideontimes$ is a removal operator satisfying **(Hyperreg)** then $\mathcal{C}(K, \divideontimes)$ satisfies (f).

PROOF. *(i)*: Suppose $(\leq, \preceq)$ satisfies (f) and suppose $\theta \notin K \divideontimes_{(\leq,\preceq)} (\theta \wedge \phi)$. Since $\divideontimes_{(\leq,\preceq)}$ satisfies **(B8)**, we already know $K \divideontimes_{(\leq,\preceq)} (\theta \wedge \phi) \subseteq K \divideontimes_{(\leq,\preceq)} \theta$, so it remains to show $K \divideontimes_{(\leq,\preceq)} \theta \subseteq K \divideontimes_{(\leq,\preceq)} (\theta \wedge \phi)$, equivalently $[K \divideontimes_{(\leq,\preceq)} (\theta \wedge \phi)] \subseteq [K \divideontimes_{(\leq,\preceq)} \theta]$. But by the proof of **(B8)** in Theorem 6, we know if $\theta \notin K \divideontimes_{(\leq,\preceq)} (\theta \wedge \phi)$ then there exists some $w_0 \in [\neg\theta] \cap \min([\neg(\theta \wedge \phi)], \leq)$. Clearly then $w_0 \in \min([\neg\theta], \leq)$. Now let $w \in [K \divideontimes_{(\leq,\preceq)} (\theta \wedge \phi)]$. Then $w \preceq w'$ for some $w' \in \min([\neg(\theta \wedge \phi)], \leq)$. Since $w' \sim w_0$ we may apply (f) to deduce $w \preceq w_0$, and so since $w_0 \in \min([\neg\theta], \leq)$ we get $w \in [K \divideontimes_{(\leq,\preceq)} \theta]$. Hence $[K \divideontimes_{(\leq,\preceq)} (\theta \wedge \phi)] \subseteq [K \divideontimes_{(\leq,\preceq)} \theta]$ as required.

(ii): Suppose $\circledast$ satisfies **(Hyperreg)** and suppose $w_1 \sim w_2$. This translates into $\neg w_1, \neg w_2 \notin K \circledast (\neg w_1 \wedge \neg w_2)$. From this we know, using **(Hyperreg)**, that $K \circledast \neg w_1$ and $K \circledast \neg w_2$ are both equal to $K \circledast (\neg w_1 \wedge \neg w_2)$. Hence for any $w_3 \in \mathcal{W}$, $\neg w_3 \notin K \circledast \neg w_1$ iff $\neg w_3 \notin K \circledast \neg w_2$, i.e., $w_3 \preceq w_1$ iff $w_3 \preceq w_2$. Thus $\mathcal{C}(K, \circledast)$ satisfies (f).

Proposition 24 $\circledast$ is a σ-liberation operator iff $\circledast = \circledast_{(\leq,\preceq)}$ for some transitive $(\leq, \preceq)$ satisfying (b) and (f).

PROOF. By results in [10] (see Corollary 3.19, p.62 there), $\circledast$ is a σ-liberation operator iff it is a linear liberation operator satisfying **(BSConserv)**. From Proposition 22 this is the same as saying $\circledast$ is a basic removal operator satisfying **(Hyperreg)** and **(BSConserv)**. The result then follows by combining Theorem 6 with Propositions 13 and 23.

Proposition 25 Let $(\leq, \preceq)$ be a transitive K-context. Then $(\leq, \preceq)$ satisfies (b) iff $(\leq, \preceq)$ satisfies (f).

PROOF. Suppose $(\leq, \preceq)$ satisfies (f) and suppose $w_1 \sim w_2$. Then since $w_1 \preceq w_1$ by reflexivity of $\preceq$, we may apply (f) to deduce $w_1 \preceq w_2$. Thus (b) holds. Note this implication (f) $\Rightarrow$ (b) holds even without assuming $(\leq, \preceq)$ is transitive. This assumption *is* required for the converse implication: Suppose $w_1 \sim w_2$ and $w_3 \preceq w_1$. From the former we get $w_1 \preceq w_2$ using (b) and so the desired $w_3 \preceq w_2$ follows from transitivity.

Proposition 26 The following are equivalent:
(i). $\circledast$ is a σ-liberation operator.
(ii). $\circledast$ is a linear liberation operator which satisfies **(BTran)**.
(iii). $\circledast$ is a basic removal operator which satisfies **(BSConserv)**.

PROOF. As stated in the text.

Proposition 27 *(i).* If $(\leq, \preceq)$ satisfies (e$'$) then $\circledast_{(\leq,\preceq)}$ satisfies **(Dichotomy)**. *(ii).* If $\circledast$ is a removal operator satisfying **(Dichotomy)** then $\mathcal{C}(K, \circledast)$ satisfies (e$'$).

PROOF. For (i), observe firstly that if $(\leq, \preceq)$ satisfies (e$'$) then for every ϕ, it follows that for every $w_1, w_2 \in [K \circledast \phi]$, $w_1 \sim w_2$. Now suppose that $(K \circledast \theta) \cup (K \circledast \phi) \nvdash \bot$. That means there is a w such that $w \in [K \circledast \theta]$ and $w \in [K \circledast \phi]$.

From the observation above it follows that for every $w' \in [K \circledast \theta]$, $w' \sim w$ and for every $w'' \in [K \circledast \phi]$, $w'' \sim w$. And so $w' \sim w''$ for every $w' \in [K \circledast \theta]$ and every $w'' \in [K \circledast \phi]$. From (e′) it then follows that $[K \circledast \theta] = [K \circledast \phi]$, which means that $K \circledast \theta = K \circledast \phi$.

For (ii), consider $\mathcal{C}(K, \circledast) = (\leq, \preceq)$ as defined in Definition 7. We need to show that $w_1 \preceq w_2$ iff $w_1 \sim w_2$. So, suppose first that $w_1 \preceq w_2$. That is, $\neg w_1 \notin K \circledast \neg w_2$. From this it follows that $w_1 \in [K \circledast \neg w_2]$. By **(B2)** it follows that $w_1 \in [K \circledast \neg w_1]$. From the combination of these two results it follows that $(K \circledast \neg w_1) \cup K \circledast w_2) \nvdash \bot$, and by **(Dichotomy)** we then have that $K \circledast \neg w_1 = K \circledast \neg w_2$. We can distinguish between two cases:

(Case 1) $K \circledast \neg w_1 = K \circledast \neg w_2 = K \circledast (\neg w_1 \wedge \neg w_2)$: In this case we immediately get that $\neg w_1 \notin K \circledast (\neg w_1 \wedge \neg w_2)$, from which it follows that $w_1 \leq w_2$, and we get that $\neg w_2 \notin K \circledast (\neg w_1 \wedge \neg w_2)$, from which it follows that $w_2 \leq w_1$. This means $w_1 \sim w_2$, which is what we wanted to prove.

(Case 2) $(K \circledast \neg w_1 = K \circledast \neg w_2) \neq K \circledast (\neg w_1 \wedge \neg w_2)$: In this case it follows by **(Dichotomy)** that $(K \circledast \neg w_1) \cup (K \circledast (\neg w_1 \wedge \neg w_2)) \vdash \bot$ and $(K \circledast \neg w_2) \cup (K \circledast (\neg w_1 \wedge \neg w_2)) \vdash \bot$. Since we know that $w_1 \in [K \circledast \neg w_1]$ and $w_2 \in [K \circledast \neg w_2]$ it follows that $w_1 \notin [K \circledast \neg w_1 \wedge \neg w_2]$ and $w_2 \notin [K \circledast \neg w_1 \wedge \neg w_2]$. So $\neg w_1 \in K \circledast (\neg w_1 \wedge \neg w_2)$ and $\neg w_2 \in K \circledast (\neg w_1 \wedge \neg w_2)$, and by **(B1)** we then have $\neg w_1 \wedge \neg w_2 \in K \circledast (\neg w_1 \wedge \neg w_2)$, which contradicts **(B2)**.

So, we have shown that in Case 1 the desired results follow, and that Case 2 cannot occur, which means we have shown that if $w_1 \preceq w_2$ then $w_1 \sim w_2$.

Now suppose that $w_1 \sim w_2$. That is, $\neg w_1 \notin K \circledast (\neg w_1 \wedge \neg w_2)$ and $\neg w_2 \notin K \circledast (\neg w_1 \wedge \neg w_2)$. So $w_1 \in [K \circledast (\neg w_1 \wedge \neg w_2)]$ and $w_2 \in [K \circledast (\neg w_1 \wedge \neg w_2)]$. By **(B2)** it also follows that $w_1 \in [K \circledast \neg w_1]$ and $w_2 \in [K \circledast \neg w_2]$. This means that $(K \circledast \neg w_1) \cup (K \circledast (\neg w_1 \wedge \neg w_2)) \nvdash \bot$ and that $(K \circledast \neg w_2) \cup (K \circledast (\neg w_1 \wedge \neg w_2)) \nvdash \bot$, and by **(Dichotomy)** it then follows that $K \circledast \neg w_1 = K \circledast (\neg w_1 \wedge \neg w_2)$ and $K \circledast \neg w_2 = K \circledast (\neg w_1 \wedge \neg w_2)$. Therefore $\neg w_1 \notin K \circledast \neg w_2$, which means that $w_1 \preceq w_2$.

Proposition 28 The following are equivalent:
(i). $\circledast$ is a dichotomous liberation operator.
(ii). $\circledast$ satisfies **(B1)–(B8)** plus **(Dichotomy)**.
(iii). $\circledast = \circledast_{(\leq, \preceq)}$ for some $(\leq, \preceq)$ which satisfies (e′).

PROOF. As stated in the text.

E Proofs for Section 6

Proposition 29 *(i).* If $(\leq, \preceq)$ satisfies (g) then $\divideontimes_{(\leq,\preceq)}$ satisfies:

(B9) If $\theta \in K \divideontimes (\theta \wedge \phi)$ then $\phi \notin K \divideontimes \theta$

(ii). If $\divideontimes$ is a removal operator satisfying **(B9)** then $\mathcal{C}(K, \divideontimes)$ satisfies (g).

PROOF. *(i).* Suppose $(\leq, \preceq)$ satisfies (g) and suppose $\theta \in K \divideontimes_{(\leq,\preceq)} (\theta \wedge \phi)$. Let $w_0 \in \min([\neg(\theta \wedge \phi)], \leq)$. Then, as we saw in the proof of **(B5)**, this means $w \in [\theta]$ for all $w \leq w_0$. This implies *(1)* $w_0 \in [\neg\phi]$ and *(2)* $w_0 < w'$ for all $w' \in \min([\neg\theta], \leq)$. Using (g), this latter implies $w_0 \preceq w'$ for all $w' \in \min([\neg\theta], \leq)$ and so in fact $w_0 \in [K \divideontimes_{(\leq,\preceq)} \theta]$. From this and *(1)* we conclude $\phi \notin K \divideontimes_{(\leq,\preceq)} \theta$.

(ii). Suppose $\divideontimes$ satisfies **(B9)** and suppose $w_1 < w_2$. Then $\neg w_2 \in K \divideontimes (\neg w_1 \wedge \neg w_2)$. Applying **(B9)** to this gives $\neg w_1 \notin K \divideontimes \neg w_2$, i.e., $w_1 \preceq w_2$ as required.

Proposition 30 *(i).* If $(\leq, \preceq)$ satisfies (h) then $\divideontimes_{(\leq,\preceq)}$ satisfies:

(B10) If $\vdash (\theta \vee \phi)$ and $\theta \notin K \divideontimes \phi$ then $\phi \in K \divideontimes (\theta \wedge \phi)$

(ii). If $\divideontimes$ is a removal operator satisfying **(B10)** then $\mathcal{C}(K, \divideontimes)$ satisfies (h).

PROOF. *(i).* Suppose $(\leq, \preceq)$ satisfies (h) and suppose $\vdash (\theta \vee \phi)$ and $\theta \notin K \divideontimes_{(\leq,\preceq)} \phi$. Then $[\neg\theta] \cap [K \divideontimes_{(\leq,\preceq)} \phi] \neq \emptyset$ so there exist $w \in [\neg\theta]$ and $w' \in \min([\neg\phi], \leq)$ such that $w \preceq w'$. Since $\vdash (\theta \vee \phi)$ and $w' \in [\neg\phi]$ we know $w' \in [\theta]$. Hence $w \neq w'$ so, by (h), $w < w'$. Now let $w_0 \in [K \divideontimes_{(\leq,\preceq)} (\theta \wedge \phi)]$. We will show $w_0 \in [\phi]$. But $w_0 \in [K \divideontimes_{(\leq,\preceq)} (\theta \wedge \phi)]$ implies $w_0 \preceq w''$ – and hence $w_0 \leq w''$ – for some $w'' \in \min([\neg(\theta \wedge \phi)], \leq)$. Since $w \in [\neg(\theta \wedge \phi)]$ this gives $w_0 \leq w$ and so, since $w < w'$, $w_0 < w'$. We conclude $w_0 \in [\phi]$ from this using $w' \in \min([\neg\phi], \leq)$. Hence we have shown $[K \divideontimes_{(\leq,\preceq)} \neg(\theta \wedge \phi)] \subseteq [\phi]$, which gives the required $\phi \in K \divideontimes_{(\leq,\preceq)} \neg(\theta \wedge \phi)$.

(ii). Suppose $\divideontimes$ satisfies **(B10)** and let $w_1 \preceq w_2$, i.e., $\neg w_1 \notin K \divideontimes \neg w_2$. If $w_1 = w_2$ we are done, so suppose $w_1 \neq w_2$. Then $\vdash (\neg w_1 \vee \neg w_2)$, so we may apply **(B10)** to deduce $\neg w_2 \in K \divideontimes (\neg w_1 \wedge \neg w_2)$, i.e., $w_1 < w_2$ as required.

Proposition 31 *(i).* If $(\leq, \preceq)$ satisfies (j) then $\divideontimes_{(\leq,\preceq)}$ satisfies:

(B11) If $\vdash (\theta \vee \phi)$ and $\theta \in K \setminus K \divideontimes \phi$ then $\phi \in K \divideontimes (\theta \wedge \phi)$

(ii). If $\divideontimes$ is a removal operator satisfying **(B11)** then $\mathcal{C}(K, \divideontimes)$ satisfies (j).

PROOF. *(i).* Suppose $\vdash (\theta \vee \phi)$ and $\theta \in K \setminus K \circledast_{(\leq,\preceq)} \phi$. From $\theta \notin K \circledast_{(\leq,\preceq)} \phi$ we know there exists $w_0 \in [\neg\theta]$ such that $w_0 \preceq w$ for some $w \in \min([\neg\phi], \leq)$. We now claim $w_0 < w$. Firstly, since $w_0 \in [\neg\theta]$ and $\vdash (\theta \vee \phi)$ we know $w_0 \in [\phi]$ and so, since $w \in [\neg\phi]$, we know $w_0 \neq w$. Secondly, since $\theta \in K$ we know $[K] \subseteq [\theta]$ and so $w_0 \notin [K]$. Applying (j) to these two with $w_0 \preceq w$ proves the desired $w_0 < w$. From this we deduce the $\leq$-minimal $\neg\theta$-worlds must be strictly below the $\leq$-minimal $\neg\phi$-worlds in $\leq$, which implies $\min([\neg(\theta \wedge \phi)], \leq) \subseteq [\phi]$. This is enough to imply $\phi \in K \circledast_{(\leq,\preceq)} (\theta \wedge \phi)$.

(ii). Suppose $\circledast$ satisfies **(B11)**. To show (j) suppose $w_1 \preceq w_2$, i.e., $\neg w_1 \notin K \circledast \neg w_2$. Suppose also $w_1 \neq w_2$ and that it is not the case that $w_1 \leq w'$ for all w', i.e., $w_1 \notin [K]$. We must show then $w_1 < w_2$. But from $w_1 \neq w_2$ we get $\vdash (\neg w_1 \vee \neg w_2)$ while from $w_1 \notin [K]$ we get $\neg w_1 \in K$. Applying **(B11)** to these two and $\neg w_1 \notin K \circledast \neg w_2$ gives $\neg w_2 \in K \circledast (\neg w_1 \wedge \neg w_2)$, i.e., $w_1 < w_2$ as required.

Proposition 32 The following are equivalent:
(i). $\dot{-}$ is a systematic withdrawal.
(ii). $\dot{-}$ satisfies **(B1)–(B8)** plus **(Vacuity)**, **(B9)** and **(B11)**.
(iii). $\dot{-} = \circledast_{(\leq,\preceq)}$ for some $(\leq, \preceq)$ which satisfies (c), (g) and (j).

PROOF. To prove *(i)*⇔*(iii)* recall from [11] that $\dot{-}$ is a systematic withdrawal iff there is some total pre-order $\leq$ over $\mathcal{W}$ such that

$$K \dot{-} \phi = K \cap Th(\nabla_{\leq}(\min([\neg\phi], \leq))),$$

where $\nabla_{\leq}(X) = \{v \mid \exists w \in X \text{ s.t. } v = w \text{ or } v < w\}$. Now observe that (c)+(g)+(j) are enough to specify $\preceq$ uniquely in terms of $\leq$ via

$$w_1 \preceq w_2 \text{ iff } w_1 \in [K] \text{ or } w_1 = w_2 \text{ or } w_1 < w_2.$$

To see this note that the left-to-right implication is exactly (j). For the converse we have $w_1 = w_2$ implies $w_1 \preceq w_2$ by reflexivity of $\preceq$ and $w_1 < w_2$ implies $w_1 \preceq w_2$ by (g). It remains to show $w_1 \in [K]$ implies $w_1 \preceq w_2$. But if $w_2 \in [K]$ then the desired conclusion follows from (c), while if $w_2 \notin [K]$ then $w_1 < w_2$ and it follows from (g). Given this, item *(iii)* of the proposition is the same as saying there exists some total pre-order $\leq$ over $\mathcal{W}$ such that

$$\begin{aligned}[K \dot{-} \phi] &= [K] \cup \{v \in \mathcal{W} \mid \exists w \in \min([\neg\phi], \leq) \text{ s.t. } v = w \text{ or } v < w\} \\ &= [K] \cup \nabla_{\leq}(\min([\neg\phi], \leq)),\end{aligned}$$

from which we can see that *(i)* and *(iii)* are saying the same thing.

The equivalence *(ii)*⇔*(iii)* follows from Theorem 6 along with Propositions 15, 29 and 31.

F Proofs for Section 7

Proposition 33 *(i).* If $\preceq$ is the equality relation then $\circledast_{(\leq,\preceq)}$ satisfies:

(B12) $\neg\phi \in K \circledast \phi$.

(ii). If $\circledast$ is a removal operator satisfying **(B12)** then $\preceq$ in $\mathcal{C}(K, \circledast)$ is the equality relation.

PROOF. *(i).* If $\preceq$ is the equality relation then clearly $[K \circledast_{(\leq,\preceq)} \phi] = \min([\neg\phi], \leq)$ and so $\neg\phi \in K \circledast_{(\leq,\preceq)} \phi$ as required.

(ii). Suppose $\circledast$ satisfies **(B12)**. To show $\preceq$ in $\mathcal{C}(K, \circledast)$ is the equality relation we need to show $\neg w_1 \notin K \circledast \neg w_2$ implies $w_1 = w_2$. But we know by **(B12)** that $[K \circledast \neg w_2] = \{w_2\}$. Hence if $w_1 \neq w_2$ then $w_1 \notin [K \circledast \neg w_2]$ which means $\neg w_1 \in K \circledast \neg w_2$ as required.

Proposition 34 If $\circledast$ is a basic removal operator then $\mathbb{R}(\circledast)$ is an AGM revision operator.

PROOF. Suppose $\circledast$ is generated by K-context $(\leq, \preceq)$. Then $K \times \phi$ is determined entirely by the total pre-order $\leq$ via $[K \times \phi] = \min([\phi], \leq)$. The fact that $\times$ is an AGM revision operator then follows from well-established results linking AGM revision with total pre-orders over the set of worlds [4,5].

Proposition 35 Let $(\leq, \preceq)$ be a K-context. Then $\preceq = \leq$ iff both (f) and (g) are satisfied.

PROOF. Let $(\leq, \preceq)$ be a K-context. Suppose $\preceq = \leq$. Then (f) reduces to the property "[$w_1 \sim w_2$ and $w_3 \leq w_1$] implies $w_3 \leq w_2$", which clearly holds by transitivity of $\leq$, while (g) reduces to the property "$w_1 < w_2$ implies $w_1 \leq w_2$", which holds trivially. Conversely suppose $(\leq, \preceq)$ satisfies both (f) and (g). We want to show $\preceq = \leq$. By definition of K-context we already have $\preceq \subseteq \leq$. For the converse inclusion suppose $w_1 \leq w_2$. If in fact $w_1 < w_2$ then we obtain the desired $w_1 \preceq w_2$ by (g). So suppose $w_1 \sim w_2$. Since $\preceq$ is reflexive we know $w_1 \preceq w_1$. Applying (f) to these two gives $w_1 \preceq w_2$ as required.

Proposition 36 $\dot{-}$ is a severe withdrawal operator iff it satisfies **(B1)–(B4)**, **(Hyperreg)** and **(B9)**.

PROOF. By results in [12] we know $\dot{-}$ is a severe withdrawal operator iff $\dot{-} = \circledast_{(\leq,\preceq)}$ where $\preceq = \leq$. Then the result follows from combining Theorem 6 and Propositions 23, 29 and 35. (Recall **(Hyperreg)** implies **(B5)–(B8)** given the fundamental rules **(B1)–(B4)**.)

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