Combining Constitutive and Regulative Norms in Input/Output Logic

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Abstract. In this paper we study three semantics to combine constitutive and regulative norms. In the first semantics, called the simple-minded semantics, the output of the constitutive norms are intermediate facts used as input for the regulative norms. The second method is called throughput, and adds the input of the constitutive norms to the intermediate facts. The third method is called reusable, because it reuses the output of the regulative norms in the input. In addition, we refine these three so-called abstract semantics such that the obligations are labeled with the intermediate facts used to derive them. These explanations in the labels can be used for argumentation, norm change or interpretation. We present complete axiomatisations for the abstract and refined version of the three semantics.

Key words: deontic logic, input/output logic, constitutive norms

1 Introduction

Constitutive norms are one of the traditional developments of normative reasoning discussed in the handbook of deontic logic [6], besides permissive norms, prima facie norms, and normative positions. They are usually contrasted with norms regulating the behavior of human beings by indicating which behaviors are obligatory, permitted and forbidden. Constitutive norms do not regulate actions or states-of-affairs, but rather they define new possible actions or states of affairs. In this paper we have little to say about constitutive norms themselves, and we refer the reader to the overview chapter of Grossi and Jones in the handbook [8]. Instead, we are interested in the *combination* of regulative and constitutive norms. As there are various ways to combine these two kinds of norms, and we believe none of them is perfect, this raises a new question. Besides choosing a logic for constitutive norms and a logic for regulative norms, the new question is:

Research question. Which combination method to choose for combining constitutive and regulative norms?

Our approach in this paper to answer this question is to define three ways of combining these two kinds of norms, and axiomatize these combinations. As always, the axiomatization presents the characteristic properties of the combination methods, which can be used to choose the method appropriate for a particular application. Moreover, we make as little commitments as possible about the representation of the norms. For example, constitutive norms are often represented by count-as conditionals "X counts as Y in the context Z" [22, 12, 9], but there does not seem to be a consensus on the representation of the context. We therefore follow Lindahl and Odelstad [15] and Boella and van der Torre [1, 2] and abstract away from the context. We thus represent constitutive norms as rules "X counts as Y." We use the general input/output logic approach [17, 18] to represent both constitutive and regulative norms, and in particular we use a 'minimal' input/output logic recently introduced by Parent and van der Torre [19].

The research question breaks down into the following four subquestions. First, how to axiomatize the simple-minded combination method of Lindahl and Odelstad [15], visualized below in Figure 1? Here C and R are the set of constitutive and regulative norms respectively. A is a set of formulas represents the facts. I is another set of formulas representing the intermediate concepts generated by the fact A and constitutive norms C. These intermediate concepts are the input of the regulative norms R. O is the output of intermediate concepts I together with regulative norms R. If we write $I(C, A)$ for the intermediate facts derived from the facts A using the constitutive norms C, and $\bigcirc(R, I)$ for the obligations derived from the intermediate facts I using the regulative norms R, then we can represent their combination method as $\ominus^*(R, C, A) = \bigcirc(R, I(C, A))$.

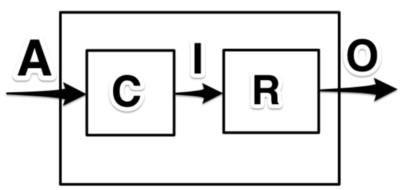

Fig. 1. Lindahl & Odelstad's combination

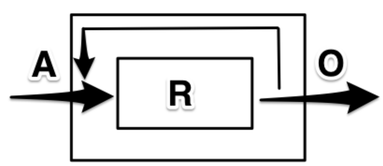

Fig. 2. Aggregative input/output logic

Second, how to relate obligations explicitly to their intermediate facts, such that these can be used for explanation tasks in argumentation, interpretation and norm change? For example, assume that a piece of paper counts as a contract, represented by the constitutive norm (*paper*,*contract*), and that the contract obliges us to pay money, represented by the regulative norm (*contract*, *pay*). From these two norms we want to derive the intermediate fact *contract* from the fact *paper*, and the obligation $pay_{contract}$, which means we are obliged to pay because there is a contract. For example, in argumentation, we can present a rebutting argument there is no obligation to pay, or an undercutting argument that there is no contract. As another example, consider the famous story of tû-tû introduced by Ross [21]. Suppose (*eat*, *tû-tû*) represents "If a person has eaten the chief's food she is tû-tû", and (*tû-tû*, *purification*) represents "If a person is tû-tû she is obligatory to be subjected to a ceremony of purification." Given the fact *eat*, from these two norms the institutional fact of *tû-tû* and the obligation $purification_{tû\text{-}tû}$, which means the person should be subjected to a ceremony of purification because she is tû-tû. Likewise, an obligation $congratulate_{checkmate}$ says that you have to congratulate your opponent because you are check mate in chess, and the obligation *takecarefamily*$_{beingmarried}$ says that you have to take care of your family because you are married.

Third, how to introduce the assumption that base facts are treated as intermediate facts too? Tosatto et al. [24] distinguish $\ominus^*(R, C, A) = \bigcirc(R, I(C, A))$ from $\ominus(R, C, A) = \bigcirc(R, A \cup I(C, A))$. For example, given $R = \{(a, x), (p, y)\}$, $C = \{(b, p)\}$ and $A = \{a, b\}$, we have $\ominus^*(R, C, A) = \{y\}$ because $I(C, A) = \{p\}$ and $\bigcirc(R, \{p\}) = \{y\}$, and $\ominus(R, C, A) = \{x, y\}$ because $A \cup I(C, A) = \{a, b, p\}$ and $\bigcirc(R, \{a, b, p\}) = \{x, y\}$.

Fourth, how to ensure that the combined system has the same properties as the individual systems? In particular, Parent and van der Torre [19] argue for a new form of deontic detachment, called aggregative deontic detachment. The corresponding rule for aggregative deontic detachment is called *aggregative cumulative transitivity* (ACT):

$$\text{(ACT)}\frac{x \in \bigcirc(R, a) \qquad y \in \bigcirc(R, a \wedge x)}{x \wedge y \in \bigcirc(R, a)}$$

In this paper we use aggregative input/output logic to represent constitutive and regulative norms. The two combinations mentioned by Tosatto et al. [24] are therefore represented by $\bigcirc(R, \bigcirc(C, A))$ and $\bigcirc(R, A \cup \bigcirc(C, A))$, where the operator $\bigcirc$ is defined in Parent and van der Torre [19]. We show in this paper that even though aggregative input/output logic satisfies ACT, both $\bigcirc(R, \bigcirc(C, A))$ and $\bigcirc(R, A \cup \bigcirc(C, A))$ do not. We therefore define a third combination of constitutive and regulative norms by forcing $\bigcirc(R, A \cup \bigcirc(C, A))$ to satisfy ACT. We call these three combinations simple-minded, throughput and reusable combination respectively. For each of the three combination, we further distinguish the abstract combination, where the output of the combination is obligation, and detailed combination, where the output of the combination is obligation together with institutional facts.

Inspired by the input/output terminology, we use the following notation. We use $\circledcirc_1$, $\circledcirc_1^+$ and $\circledcirc_3^+$ for the simple-minded, throughput and reusable abstract combination respectively. For detailed combination, we use $\odot_1$, $\odot_1^+$ and $\odot_3^+$.

The layout of this paper is as follows. We survey aggregative input/output logic in Section 2. Then we introduce basic, throughput and reusable combinations in Section 3 to 5 respectively. We then discuss related work and future research. We finish this paper in Section 8.

2 Aggregative input/output logic

Parent and van der Torre [19] introduce aggregative input/output logic, based on the following ideas. On the one hand, deontic detachment (DD) or cumulative transitivity (CT) is fully in line with the tradition in deontic logic. For instance, the Danielsson-Hansson-Lewis semantics [5, 10, 14] for conditional obligation validates such a law. On the other hand, they also observe that potential counterexamples to DD may be found in the literature. Parent and van der Torre illustrate this with the following example, due to Broome [4, §7.4]:

You ought to exercise hard everyday
If you exercise hard everyday, you ought to eat heartily
?⋆ You ought to eat heartily

Intuitively, the obligation to eat heartily no longer holds, if you take no exercise. Like the others, Parent and van der Torre claim that this counterexample suggests an alternative (they call it "aggregative") form of detachment, which keeps track of what has been previously detached. They therefore reject the CT rule, and they accept the weaker ACT rule. As a consequence, and following amongst others [11, 7, 25, 23], weakening the output is no longer accepted either.

Let $\mathbb{P} = \{p_0, p_1, \ldots\}$ be a countable set of propositional letters and L be the propositional logic built upon $\mathbb{P}$. We write $\phi \dashv\vdash \psi$ for logical equivalence in the logic L. Let R be a set of ordered pairs of formulas of L. A pair $(a, x) \in R$, call it a regulative norm, is read as "given a, it ought to be x". Let $A \subseteq L$ be a set of formulas, $R(A) = \{x \in L | (a, x) \in R, a \in A\}$ be set theoretically understood as the image of A under function R. The semantics of aggregative input/output logic is defined as following:

Definition 1. (aggregative input/output logic [19]) *For every $R \subseteq L \times L$, $A \subseteq L$, $x \in \bigcirc(R, A)$ iff there is finite $R' \subseteq R$ with $R'(A) \neq \emptyset$ such that $\forall B = Cn(B)$, if $A \cup R'(B) \subseteq B$ then $x \dashv\vdash \bigwedge R'(B)$.*

The above definition can be visualized by Figure 2. In the definition there is a qualification over a logically closed set B, which represents the input of R. B is required to extend $A \cup R'(B)$ because the left arrow is labeled by A and there is influx from the right arrow (representing $R'(B)$) to the left arrow.

The following example illustrates aggregative deontic detachment.

Example 1. Let regulative norms be $R = \{(a, x), (a \wedge x, y)\}$ and input $A = \{a\}$. We have $\bigcirc(R, A) = \{x,\ x \wedge y, \ldots\}$. The table below illustrates how to calculate $\bigcirc(R, A)$, where B^* is the smallest set of formulas such that we have $B^* = Cn(B^*)$ and $A \cup R'(B^*) \subseteq B^*$. We can then derive the obligation for $x \wedge y$, but not an obligation for y, which means ACT hold in aggregative input/output logic but CT does not.

A	R'	B^*	$R'(B^*)$	$\bigwedge R'(B^*)$
$\{a\}$	$\{(a, x)\}$	$Cn(\{a, x\})$	$\{x\}$	$\{x, \ldots\}$
{a}	R	$Cn(\{a, x, y\})$	$\{x, y\}$	$\{x \wedge y, \ldots\}$

The proof system contains three rules: strengthening of the antecedent (SI), output equivalence (OEQ) and aggregative cumulative transitivity (ACT).

Definition 2. (Proof system of aggregative input/output logic [19]) *Let $D(R)$ be the smallest set of arguments such that $R \subseteq D(R)$ and $D(R)$ is closed under the following rules:*

- *SI: from (a, x) and $b \vdash a$ to (b, x)*
- *OEQ: from (a, x) and $x \dashv\vdash y$ to (a, y)*
- *ACT: from (a, x) and $(a \wedge x, y)$ to $(a, x \wedge y)$.*

The rule AND is derivable in aggregative input/output logic.

- AND: from (a, x) and (a, y) to $(a, x \wedge y)$

Parent and van der Torre define $x \in D(R, A)$ iff there exist $a_1, \ldots, a_n \in A$ such that $(a_1 \wedge \ldots \wedge a_n, x) \in D(R)$. The following completeness result is proved [19].

Theorem 1. **(Completeness of aggregative input/output logic [19])** *Given an arbitrary normative system R and a set A of formulas, $D(R, A) = \bigcirc(N, A)$.*

3 Simple-minded combination

The idea of simple-minded combination is illustrated by Figure 1. There is a set of constitutive norms $C \subseteq L \times L$ and a set of regulative norms $R \subseteq L \times L$. The input is a set A of formulas representing facts. $I = \bigcirc(C, A)$ is the output produced by the semantics of aggregative input/output logic given C and A. I is understood as the intermediate facts and further used as the input to regulative norms R to generate obligations $O = \bigcirc(R, I)$.

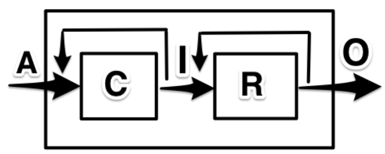

Fig. 3. Basic combination

We use aggregative input/output logic as our tool to analyze both constitutive and regulative norms. Since aggregative input/output logic is reusable in the sense its output can be reused as input, we represent basic combination by Figure 3 with arrows representing reusability.

3.1 Simple-minded abstract combination

Simple-minded abstract combination can be built straightforwardly by a composition of two aggregative input/output logic.

Definition 3. *Let $C, R \subseteq L \times L, A \subseteq L$, the semantics of simple-minded abstract combination is:*

$$\circledcirc_1(C, R, A) = \bigcirc(R, \bigcirc(C, A)).$$

We use the following example to illustrate simple-minded abstract combination:

Example 2. Let $A = \{$*eat*$\}, C = \{($*eat*, *tû-tû*$)\}, R = \{($*tû-tû*, *purification*$)$, $($*eat*, *sorry*$)\}$. Here (*eat*, *sorry*) means "if a person has eaten the chief's food she should say sorry." Then we have $\bigcirc(C, A) = \{$*tû-tû*,$\ldots\}$, $\circledcirc_1(C, R, A) = \{$*purification*,$\ldots\}$. Note we do not have *sorry* $\in \circledcirc_1(C, R, A)$ because *eat* $\notin \bigcirc(C, A)$.

The proof system of simple-minded abstract combination is defined as follows:

Definition 4. *Let $C, R \subseteq L \times L$, the proof system of simple-minded abstract combination is defined as follows:*

$$\text{Đ}_1(C,R) = \{(a,x) \mid \textit{there is } p \in L \textit{ such that } (a,p) \in D(C) \textit{ and } (p,x) \in D(R)\}$$

We call the rule to derives $(a,x) \in \text{Đ}_1(C,R)$ from $(a,p) \in D(C)$ and $(p,x) \in D(R)$ constitutive/regulative transitivity (CR-T). The following is an example to illustrate the proof system of simple-minded abstract combination:

Example 3. From $C = \{(a,x),(a \wedge x, y)\}$, $R = \{(y,z)\}$ we can derive $(a,z) \in \text{Đ}_1(C,R)$ as following:

$$\dfrac{\dfrac{(a,x)\in C \quad (a\wedge x)\in C}{(a,x\wedge y)\in D(C)}(\text{ACT}) \quad \dfrac{(y,z)\in R}{(x\wedge y,z)\in D(R)}(\text{SI})}{(a,z) \in \text{Đ}_1(C,R)}(\text{CR-T})$$

The semantics and proof theory of simple-minded abstract combination are connected by the following completeness result:

Theorem 2 (Completeness of simple-minded abstract combination). *For two sets $C, R \subseteq L \times L$, we have*

$x \in \circledcirc_1(C,R,a)$ *iff* $(a,x) \in \text{Đ}_1(C,R)$.[1]

Proof. (sketch) Assume $x \in \circledcirc_1(C,R,a)$, then $x \in \bigcirc(R, \bigcirc(C,a))$. By Theorem 1 we know there exist $p_1,\ldots,p_n \in \bigcirc(C,a)$ such that $x \in \bigcirc(R,\{p_1,\ldots,p_n\})$. Then we can further deduce $(a, p_1 \wedge \ldots \wedge p_n) \in D(C)$ and $(p_1 \wedge \ldots \wedge p_n, x) \in D(R)$. Therefore $(a,x) \in \text{Đ}_1(C,R)$.

The other direction is easier and left to the reader. □

It can be verified that the simple-minded abstract combination satisfies SI, OEQ and AND.

Proposition 1. *SI, OEQ and AND are valid in simple-minded abstract combination.*

Proof. The proof is relatively easy and safely left to the reader. □

3.2 Simple-minded detailed combination

For detailed combinations, the tricky thing is we want to produce not obligation, but obligation with institutional facts. Formally, the output for detailed combination is of the form x_p, where $x, p \in L$. Such output has never been defined in the input/output logic literature. Technically, the semantics of simple-minded detailed combination is defined as follows, in a flavor similar to aggregative input/output logic:

Definition 5. *Let $C, R \subseteq L \times L, A \subseteq L$, we define $x_p \in \odot_1(C,R,A)$ iff there exist finite $C' \subseteq C, R' \subseteq R$ such that $\forall B_1 = Cn(B_1)$, $B_2 = Cn(B_2)$, if $A \cup C'(B_1) \subseteq B_1$ then $p \dashv\vdash \bigwedge C'(B_1)$, if $C'(B_1) \cup R'(B_2) \subseteq B_2$, then $x \dashv\vdash \bigwedge R'(B_2)$.*

[1] The proofs of theorems of this paper can be found in a technical report on the first author's website.

In the semantics of aggregative input/output logic (Definition 2), we pick a set R' of the norms and qualify over a set of formulas B, which is closed under logic consequence. In the semantics of basic detailed combination, we pick two sets C' and R', and we qualify over two sets of formulas B_1, B_2, which are both closed under logic consequence.

The set B_1 is the input for C'. As it is shown in Figure 3 , we require it to extend $A \cup C'(B_1)$ because there is an arrow labelled A inject to C' and there is another arrow, the arrow from I to A, also inject to C'. Here note that I is understood as $C'(B_1)$.

Similarly, the set B_2 is the input for R'. We require it to contain $C'(B_1) \cup R'(B_2)$ because there is an arrow labelled I inject to R and there is another arrow, the arrow from O to I, also inject to R. Here note that O can be understood as $R'(B_2)$.

Definition 6 (Proof theory of simple-minded detailed combination). *For two sets $C, R \subseteq L \times L$, the proof theory of simple-minded detailed combination is:*

$$\mathbf{D}_1(C, R) = \{(a, x_p) \mid \textit{there is } p \in L \textit{ such that } (a, p) \in D(C) \textit{ and } (p, x) \in D(R)\}$$

We call the rule which derives $(a, x_p) \in \mathbf{D}_1(C, R)$ from $(a, p) \in D(C)$ and $(p, x) \in D(R)$ constitutive/regulative transitivity (CR-T$'$).

Example 4. (continued) From $C = \{(a, x), (a \wedge x, y)\}$, $R = \{(y, z)\}$ we can derive $(a, z_{x \wedge y}) \in \mathbf{D}_1(C, R)$ as following:

$$\dfrac{\dfrac{(a,x) \in C \quad (a \wedge x) \in C}{(a, x \wedge y) \in D(C)}(\text{ACT}) \quad \dfrac{(y,z) \in R}{(x \wedge y, z) \in D(R)}(\text{SI})}{(a, z_{x \wedge y}) \in \mathbf{D}_1(C, R)}(\text{CR-T}')$$

The above proof theory relies heavily on the proof theory of aggregative input/output logic. Moreover, it works separately on constitutive and regulative norms and combine them together at the last step by the CR-T$'$ rule. We alternatively define an equivalent proof theory more directly on expressions of the form (a, x_p).

Definition 7. *Given C, R, let $\mathbf{D}'_1(C, R)$ be the smallest set of arguments such that $(\top, \top_\top) \in \mathbf{D}'_1(C, R)$, $\{(a, \top_p) \mid (a, p) \in C\} \subseteq \mathbf{D}'_1(C, R)$ and $\mathbf{D}'_1(C, R)$ is closed under the following rules:*

- *SI: strengthening of the input: from (a, x_p) to (b, x_p) whenever $b \vdash a$*
- *IOEQ: intermediate and output equivalence: from (a, x_p) to (a, y_q) whenever $p \dashv\vdash q$ and $x \dashv\vdash y$*
- *ACTI: aggregative cumulative transitivity for the intermediate: from (a, x_p) and $(a \wedge p, x_q)$ to $(a, x_{p \wedge q})$*
- *ACTO: aggregative cumulative transitivity for output: from (a, x_p) and $(a \wedge x, y_p)$ to $(a, x \wedge y_p)$,*

and the following indexed constitutive/regulative transitivity (ICR-T) rule:

- *if $(a, \top_p) \in \mathbf{D}'_1(C, R)$ and $(p, x) \in D(R)$ then $(a, x_p) \in \mathbf{D}'_1(C, R)$.*

Example 5. (continued) Given $C = \{(a, x), (a \wedge x, y)\}$, $R = \{(y, z)\}$, we first derive $(a, \top_x)$ and $(a \wedge x, \top_y)$, then we derive $(a, z_{x \wedge y}) \in \mathbf{D}_1(C, R)$ as following:

$$\dfrac{\dfrac{(a, \top_x) \quad (a \wedge x, \top_y)}{(a, \top_{x \wedge y})}(\text{ACTI}) \quad \dfrac{(y,z) \in R}{(x \wedge y, z) \in D(R)}(\text{SI})}{(a, z_{x \wedge y})}(\text{ICR-T})$$

The proof theory $\mathbf{D}'_1(C,R)$ may look unusual at first glance, but it resembles the proof theory of aggregative input/output logic. They both contain rule like strengthening of the input, output equivalence and aggregative cumulative transitivity. The equivalence of $\mathbf{D}_1(C,R)$ and $\mathbf{D}'_1(C,R)$ is stated by the following proposition:

Proposition 2. *For all $C,R \subseteq L \times L$, $(a,x_p) \in \mathbf{D}_1(C,R)$ iff $(a,x_p) \in \mathbf{D}'_1(C,R)$.*

Proof. ($\Rightarrow$) Assume $(a,x_p) \in \mathbf{D}_1(C,R)$. Then $(a,p) \in D(C)$ and $(p,x) \in D(R)$. By Lemma 1 we know $(a,\top_p) \in \mathbf{D}'_1(C,R)$. Then by the ICR-T rule we know $(a,x_p) \in \mathbf{D}'_1(C,R)$.

($\Leftarrow$) Assume $(a,x_p) \in \mathbf{D}'_1(C,R)$. We prove by induction on the length of derivation. We have the following cases: Base cases:

- (a,x_p) is $(\top,\top_\top)$. Since we have $(\top,\top) \in D(C)$, $(\top,\top) \in D(R)$, we have $(\top,\top_\top) \in \mathbf{D}_1(C,R)$.
- (a,x_p) is $(a,\top_p)$ and it is derived by $(a,p) \in C$. Then we know $(a,p) \in D(C)$. More over, from $(\top,\top) \in D(R)$ and using SI rule we have $(p,\top) \in D(R)$. Then we have $(a,\top_p) \in \mathbf{D}_1(C,R)$.

Inductive cases:

- (a,x_p) is derived from $(a,\top_p) \in \mathbf{D}'_1(C,R)$ and $(p,x) \in D(R)$. Then by induction hypothesis we have $(a,p) \in D(C)$ and $(p,\top) \in D(R)$. From $(a,p) \in D(C)$ and $(p,x) \in D(R)$ we know$(a,x_p) \in \mathbf{D}_1(C,R)$.
- (a,x_p) is derived by using SI rule. Then there exist $(b,x_p) \in \mathbf{D}'_1(C,R)$ and $a \vdash b$. Then by induction hypothesis we have $(b,p) \in D(C)$ and $(p,x) \in D(R)$. From $a \vdash b$ and $(b,p) \in D(C)$ we know $(a,p) \in D(C)$. This plus $(p,x) \in D(R)$ we can derive $(a,x_p) \in \mathbf{D}_1(C,R)$.
- (a,x_p) is derived by using IOEQ rule. Then there exist $(a,y_q) \in \mathbf{D}'_1(C,R)$ such that $x \dashv\vdash y$ and $p \dashv\vdash q$. Then by induction hypothesis we have $(a,y) \in D(C)$, $(y,q) \in D(R)$. Now by applying OEQ rule to $D(C)$ we have $(a,x) \in D(C)$. By applying SI rule to $D(R)$ we have $(x,q) \in D(R)$, and by OEQ $(x,p) \in D(R)$. Now we can derive $(a,x_p) \in \mathbf{D}'_1(C,R)$.
- $(a,x_{p\wedge q})$ is derived the ACTI rule. Then there exist $(a,x_p),(a\wedge p,x_q) \in \mathbf{D}'_1(C,R)$. By induction hypothesis we know $(a,p) \in D(C)$, $(p,x) \in D(R)$, $(a \wedge p,q) \in D(C)$, $(q,x) \in D(R)$. Applying ACT to $(a,p) \in D(C)$ and $(a \wedge p,q) \in D(C)$ we can derive $(a,p \wedge q) \in D(C)$. Applying SI to $(q,x) \in D(R)$ we can derive $(p \wedge q,x) \in D(R)$. Therefore $(a,x_{p\wedge q}) \in \mathbf{D}_1(C,R)$.
- $(a,x \wedge y_p)$ is derived the ACTO rule. Then there exist $(a,x_p),(a \wedge x,y_p) \in \mathbf{D}'_1(C,R)$. By induction hypothesis we know $(a,p) \in D(C)$, $(p,x) \in D(R)$, $(a \wedge x,p) \in D(C)$, $(p,y) \in D(R)$. Applying AND rule to $D(R)$ we have $(p,x \wedge y) \in D(R)$. Therefore $(a,x \wedge y_p) \in \mathbf{D}_1(C,R)$.

Lemma 1. *Given an arbitrary constitutive normative system C, if $(a,p) \in D(C)$ then $(a,\top_p) \in \mathbf{D}'_1(C,R)$.*

Proof. (sketch) If $(a, p) \in C$, then by the definition of $\mathbf{D}'_1(C, R)$ we know $(a, \top_p) \in \mathbf{D}'_1(C, R)$.
If $(a, p) \in D(C)$ is derived by SI, then there exist $(b, p) \in D(C)$, $a \vdash b$. By induction hypothesis we know $(b, \top_p) \in \mathbf{D}'_1(C, R)$. Therefore by SI we know $(a, \top_p) \in \mathbf{D}'_1(C, R)$.
If (a, p) is derived by OEQ, then there exist $(a, q) \in D(C)$, $p \vdash q$. By induction hypothesis we know $(a, \top_q) \in \mathbf{D}'_1(C, R)$. Therefore by IOEQ we know $(a, \top_p) \in \mathbf{D}'_1(C, R)$.
If $(a, p \wedge q)$ is derived by ACT, then there exist $(a, p), (a \wedge p, q) \in D(C)$. By induction hypothesis we know $(a, \top_p), (a \wedge p, \top_q) \in \mathbf{D}'_1(C, R)$. Therefore by ACTI we know $(a, \top_{p \wedge q}) \in \mathbf{D}'_1(C, R)$. $\square$

With the proof theory and semantics as defined, we have the following completeness result.

Theorem 3 (Completeness of basic detailed combination). *For all* $C, R \subseteq L \times L$, $(a, x_p) \in \mathbf{D}_1(C, R)$ *iff* $x_p \in \odot_1(C, R, a)$.

(soundness) The soundness is relative easy to prove. Here we skip it.
(Completeness) Assume $(a, x_p) \in \mathbf{D}_1(C, R)$, then $(a, p) \in D(C), (p, x) \in D(R)$. Then by the Theorem 1 we know $p \in \bigcirc(C, a)$ and $x \in \bigcirc(R, p)$. By the semantics of aggregative input/output logic we can check that $x_p \in \odot_1(C, R, a)$. The process is routine, here we skip it.

4 Throughput combination

In this section strengthen simple-minded combination to throughput combination such that the fact A can directly be used by regulative norms R, see Figure 4.

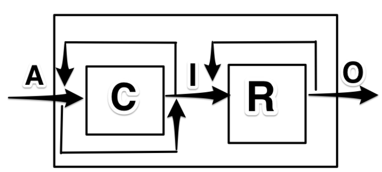

Fig. 4. Throughput combination

4.1 Throughput abstract combination

Throughput abstract combination can be built with the help of Figure 4, where both A and the output $\bigcirc(C, A)$ are part of the input of R.

Definition 8 (Semantics of throughput abstract combination). *Let* $C, R \subseteq L \times L$, $A \subseteq L$, $\circledcirc_1^+(C, R, A) = \bigcirc(R, A \cup \bigcirc(C, A))$.

The following is the proof system:

Definition 9 (Proof system of throughput abstract combination). *Let $C, R \subseteq L \times L$, the proof system of throughput abstract combination is:*
$\mathfrak{D}_1^+(C, R) = \{(a, x) \mid$ *there is* $p \in L$ *such that* $(a, p) \in D(C)$ *and* $(a \wedge p, x) \in D(R)\}$

We call the rule to derives $(a, x) \in \mathfrak{D}_1(C, R)$ from $(a, p) \in D(C)$ and $(a \wedge p, x) \in D(R)$ constitutive/regulative cumulative transitivity (CR-CT).

The semantics and proof theory of basic abstract combination are connected by the following completeness result:

Theorem 4 (Completeness of basic abstract combination). *Let $C, R \subseteq L \times L$, we have $x \in \circledcirc_1^+(C, R, a)$ iff $(a, x) \in \mathfrak{D}_1^+(C, R)$.*

Proof. The proof is similar to the proof of Theorem 2, here we skip it. □

4.2 Throughput detailed combination

In parallel to the simple-minded detailed combination, we introduce the semantics and proof theory of throughput detailed combination. The semantics of simple-minded detailed combination is similar to the semantics of aggregative input/output logic.

Definition 10. *Let $C, R \subseteq L \times L, A \subseteq L$, we define $x_p \in \odot_1^+(C, R, A)$ iff there exist finite $C' \subseteq C, R' \subseteq R$ such that $\forall B_1 = Cn(B_1)$ and $\forall B_2 = Cn(B_2)$, if $A \cup C'(B_1) \subseteq B_1$ then $p \dashv\vdash \bigwedge C'(B_1)$, if $A \cup C'(B_1) \cup R'(B_2) \subseteq B_2$, then $x \dashv\vdash \bigwedge R'(B_2)$*

Like the semantics of simple-minded detailed combination, here we pick two set C' and R', and we qualify over two sets of formulas B_1, B_2, which are both closed under logic consequence. The only difference is: for B_2, here we require it to extend A, while in simple-minded detailed combination we don't have such requirements. The reason of such difference can be detected by comparing Figure 3 and 4. In Figure 4 there is an arrow from A to I, while in Figure 3 there is not.

Definition 11 (Proof system of throughput detailed combination). *Given two sets $C, R \subseteq L \times L$, the proof system of throughput detailed combination is:*
$\mathbf{D}_1^+(C, R) = \{(a, x_p) \mid$ *there is* $p \in L$ *such that* $(a, p) \in D(C)$ *and* $(a \wedge p, x) \in D(R)\}$

As the proof theory $\mathbf{D}_1$, $\mathbf{D}_1^+$ heavily relies on the proof theory of aggregative input/output logic. A more independent proof theory is defined as follows:

Definition 12. *Let $\mathbf{D}_1^{+'}(C, R)$ be the smallest set such that $(\top, \top_\top) \in \mathbf{D}_1^{+'}(C, R)$, $\{(a, \top_p) \mid (a, p) \in C\} \subseteq \mathbf{D}_1^{+'}(C, R)$, $\{(a, x_\top) \mid (a, x) \in R\} \subseteq \mathbf{D}_1^{+'}(C, R)$ and $\mathbf{D}_1^{+'}(C, R)$ is closed under the rules SI, IOEQ, ACTI, ACTO and the following rule:*

- *if $(a, \top_p) \in \mathbf{D}_1^{+'}(C, R)$ and $(a \wedge p, x) \in D(R)$ then $(a, x_p) \in \mathbf{D}_1^{+'}(C, R)$*

One difference between $\mathbf{D}_1^{+'}(C, R)$ and $\mathbf{D}'_1(C, R)$ is: for $\mathbf{D}_1^{+'}(C, R)$ we require it to extend $\{(a, x_\top) \mid (a, x) \in R\}$. This feature in some sense reveals that regulative norms can be derived directly in throughput combination.

The equivalence of $\mathbf{D}_1^+$ and $\mathbf{D}_1^{+'}$ is stated in the following proposition.

Proposition 3. *For all $C, R \subseteq L \times L$, $(a, x_p) \in \mathbf{D}_1^{+}(C, R)$ iff $(a, x_p) \in \mathbf{D}_1^{+'}(C, R)$*

Proof. Assume $(a, x_p) \in \mathbf{D}_1^{+}(C, R)$. Then $(a, p) \in D(C)$ and $(a \wedge p, x) \in D(R)$. By Lemma 2 we know $(a, \top_p) \in \mathbf{D}_1^{+'}(C, R)$. From $(a, \top_p) \in \mathbf{D}_1^{+'}(C, R)$ and $(a \wedge p, x) \in D(R)$ we derive $(a, x_p) \in \mathbf{D}_1^{+'}(C, R)$.

Assume $(a, x_p) \in \mathbf{D}_1^{+'}(C, R)$. We prove by induction on the length of derivation. W have the following cases:
Base cases:

- If (a, x_p) is $(\top, \top_\top)$. Since $(\top, \top) \in D(C), D(R)$, we know $(\top \wedge \top, \top) \in D(R)$, therefore $(\top, \top_\top) \in \mathbf{D}_1^{+}(C, R)$.
- If $(a, x_\top) \in \mathbf{D}_1^{+'}(C, R)$ and it is derived by $(a, x) \in R$. Then we have $(a \wedge \top, x) \in D(R)$. From $(\top, \top) \in D(C)$ we can use SI to derive $(a, \top) \in D(C)$. From $(a, \top) \in D(C)$ and $(a \wedge \top, x) \in D(R)$ we know $(a, x_\top) \in \mathbf{D}_1^{+}(C, R)$.
- If $(a, \top_p) \in \mathbf{D}_1^{+'}(C, R)$ and it is derived by $(a, p) \in C$. From $(\top, \top) \in D(R)$ we can use SI to derive $(a \wedge p, \top) \in D(R)$. Therefore $(a, \top_p) \in \mathbf{D}_1^{+}(C, R)$.

Inductive steps:

- if $(a, x_p) \in \mathbf{D}_1^{+'}(C, R)$ and it is derived by SI. Then there exist b such that $a \vdash b$ and $(b, x_p) \in \mathbf{D}_1^{+'}(C, R)$. By induction hypothesis we have $(b, x_p) \in \mathbf{D}_1^{+}(C, R)$. Therefore $(b, p) \in D(C)$ and $(b \wedge p, x) \in D(R)$. By $a \vdash b$ we can deduce $(a, p) \in D(C)$ and $(a \wedge p, x) \in D(R)$. Therefore $(a, x_p) \in \mathbf{D}_1^{+}(C, R)$.
- if $(a, x_p) \in \mathbf{D}_1^{+'}(C, R)$ and it is derived by IOEQ. Then there exist q, y such that $x \dashv\vdash y$ and $p \dashv\vdash q$ and $(a, y_q) \in \mathbf{D}_1^{+'}(C, R)$. By induction hypothesis we have $(a, y_q) \in \mathbf{D}_1^{+}(C, R)$. Therefore $(a, q) \in D(C)$ and $(a \wedge q, y) \in D(R)$. By $p \dashv\vdash q$ and $x \dashv\vdash y$ we can deduce $(a, p) \in D(C)$ by OEQ and $(a \wedge p, x) \in D(R)$ by SI and OEQ. Therefore $(a, x_p) \in \mathbf{D}_1^{+}(C, R)$.
- if $(a, x_{p \wedge q}) \in \mathbf{D}_1^{+'}(C, R)$ and it is derived by ACTI. Then $(a, x_p) \in \mathbf{D}_1^{+'}(C, R)$ and $(a \wedge p, x_q) \in \mathbf{D}_1^{+'}(C, R)$. By induction hypothesis we have $(a, x_p) \in \mathbf{D}_1^{+}(C, R)$ and $(a \wedge p, x_q) \in \mathbf{D}_1^{+}(C, R)$. Therefore $(a, p) \in D(C)$, $(a \wedge p, x) \in D(R)$, $(a \wedge p, q) \in D(C)$ and $(a \wedge p \wedge q, x) \in D(R)$. Therefore by ACT we have $(a, p \wedge q) \in D(C)$. This plus $(a \wedge p \wedge q, x) \in D(R)$ we derive $(a, x_{p \wedge q}) \in \mathbf{D}_1^{+}(C, R)$.
- if $(a, x \wedge y_p) \in \mathbf{D}_1^{+'}(C, R)$ and it is derived by ACTO. Then $(a, x_p) \in \mathbf{D}_1^{+'}(C, R)$ and $(a \wedge x, y_p) \in \mathbf{D}_1^{+'}(C, R)$. By induction hypothesis we have $(a, x_p) \in \mathbf{D}_1^{+}(C, R)$ and $(a \wedge x, y_p) \in \mathbf{D}_1^{+}(C, R)$. Therefore $(a, p) \in D(C)$, $(a \wedge p, x) \in D(R)$, $(a \wedge x, p) \in D(C)$ and $(a \wedge x \wedge p, y) \in D(R)$. Therefore by ACT we have $(a \wedge p, x \wedge y) \in D(R)$. This plus $(a, p) \in D(C)$ we can derive $(a, x \wedge y_p) \in \mathbf{D}_1^{+}(C, R)$.
- (a, x_p) is derived from $(a, \top_p) \in \mathbf{D}_1^{+'}(C, R)$ and $(a \wedge p, x) \in D(R)$. Then by induction hypothesis we have $(a, p) \in D(C)$ and $(a \wedge p, \top) \in D(R)$. From $(a, p) \in D(C)$ and $(a \wedge p, x) \in D(R)$ we know$(a, x_p) \in \mathbf{D}_1^{+}(C, R)$. $\square$

Lemma 2. *Given an arbitrary constitutive normative system C, if $(a, p) \in D(C)$ then $(a, \top_p) \in \mathbf{D}_1^{+'}(C, R)$.*

Proof. (sketch) If $(a, p) \in C$, then by the definition of $\mathbf{D}_1^{+'}(C, R)$ we know $(a, \top_p) \in \mathbf{D}_1^{+'}(C, R)$.
If $(a, p) \in D(C)$ is derived by SI, then there exist $(b, p) \in D(C)$, $a \vdash b$. By induction hypothesis we know $(b, \top_p) \in \mathbf{D}_1^{+'}(C, R)$. Therefore by SI we know $(a, \top_p) \in \mathbf{D}_1^{+'}(C, R)$.
If (a, p) is derived by OEQ, then there exist $(a, q) \in D(C)$, $p \vdash q$. By induction hypothesis we know $(a, \top_q) \in \mathbf{D}_1^{+'}(C, R)$. Therefore by IOEQ we know $(a, \top_p) \in \mathbf{D}_1^{+'}(C, R)$.
If $(a, p \wedge q)$ is derived by ACT, then there exist $(a, q), (a \wedge p, q) \in D(C)$. By induction hypothesis we know $(a, \top_p), (a \wedge p, \top_q) \in \mathbf{D}_1^{+'}(C, R)$. Therefore by ACTI we know $(a, \top_{p \wedge q}) \in \mathbf{D}_1^{+'}(C, R)$. $\square$

With the proof theory and semantics as defined, we have the following completeness result.

Theorem 5 (Completeness of throughput detailed combination). *For all $C, R \subseteq L \times L$, $(a, x_p) \in \mathbf{D}_1^{+}(C, R)$ iff $x_p \in \odot_1^{+}(C, R, a)$.*

Proof. (soundness) For soundness the proof is similar to the soundness part of Theorem 2. Here we skip it.
(Completeness) Assume $(a, x_p) \in \mathbf{D}_1^{+}(C, R)$. Then by the completeness of aggregative input/output logic we know $p \in \bigcirc(C, a)$ and $x \in \bigcirc(R, a \wedge p)$. By the semantics of aggregative input/output logic we can check that $x_p \in \odot_1^{+}(C, R, a)$. The process is routine, here we skip it. $\square$

5 Reusable combination

Now we turn to reusable combinations. As illustrated by the arrow from O to A in Figure 5, reusable combination is the extension of throughput combination which allows the output of regulative norms to be reused as input for constitutive norms. In this case the input of C have three resource: the arrow A, the arrow from I to A, and the arrow from O to A. The input of R have exactly the same resource. Therefore we can change Figure 5 to Figure 6 such that C and R have the same input.

5.1 Reusable abstract combination

The fact that reusable combination is the extension of throughput combination which allows the reusability of output from R suggests that reusable combination can be defined as the extension of throughput combination which validates the ACT rule. While the proof theory of reusable abstract combination is a straightforward extension of its throughput companion, its semantics is more complex.

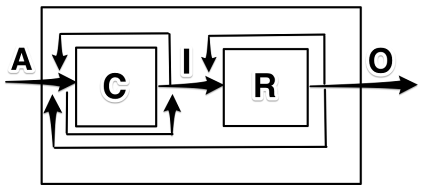

Fig. 5. Reusable combination

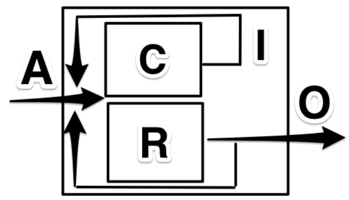

Fig. 6. Serial visualization

Definition 13 (Semantics of reusable abstract combination). *Let $C, R \subseteq L \times L$, and $A \subseteq L$, we define $x \in \circledcirc_3^+(C, R, A)$ iff there exist finite $C' \subseteq C, R' \subseteq R$ such that $\forall B = Cn(B)$: if $A \cup C'(B) \cup R'(B) \subseteq B$ then $x \dashv\vdash \bigwedge R'(B)$.*

Definition 14 (Proof system of reusable abstract combination). *Let $C, R \subseteq L \times L$, the proof system of reusable abstract combination is defined as follows:*
$Đ_3^+(C, R)$ is the smallest set such that $Đ_1^+(C, R)(C, R) \subseteq Đ_3^+(C, R)$ and $Đ_3^+(C, R)$ is closed under the ACT rule.

The above semantics reflects the ideas illustrated by Figure 6. We qualify over a set B, which is used as the input for C and R. Such B is an extension of A, I (representing $C(B)$) and O (representing $R(B)$). That's why we require $A \cup C'(B) \cup R'(B) \subseteq B$.

The semantics and proof theory of basic abstract combination are connected by the following completeness result:

Theorem 6 (Completeness of basic abstract combination). *Let $C, R \subseteq L \times L$, we have $x \in \circledcirc_3^+(C, R, a)$ iff $(a, x) \in Đ_3^+(C, R)$.*

Proof. The proof is covered by the proof of Theorem 7, here we skip it.

5.2 Reusable detailed combination

The semantics of reusable detailed combination is an extension of its abstract companion.

Definition 15 (Semantics of reusable detailed combination). *Let $C, R \subseteq L \times L$, and $A \subseteq L$, we define $x_p \in \odot_3^+(C, R, A)$ iff there exist finite $C' \subseteq C, R' \subseteq R$ such that $\forall B = Cn(B)$: if $A \cup C'(B) \cup R'(B) \subseteq B$ then $p \dashv\vdash \bigwedge C'(B), x \dashv\vdash \bigwedge R'(B)$.*

The following example is a variant from Makinson [16] to illustrate reusable combination.

Example 6. Let $C = \{(A4$, size),(t25×15, area)$\}$, $R = \{$(size, t25×15), (area, ref10)$\}$. Here "A4" means "the paper is an A4 paper", "size" means "It is of standard size", "t25×15" means "the text area is 25 by 15 cm", "area" means "It is of standard text area", "ref10" means "the font size for the reference is 10 points." In this setting we have $\circledcirc_3^+(C, R, \{\text{A4}\}) = \{\top$, t25×15, t25×15 $\wedge$ ref10, $\ldots\}$, $\odot_3^+(C, R, \{\text{A4}\}) = \{\top_\top$, t25×15$_{\text{size}}$, t25×15 $\wedge$ ref10$_{\text{size}\wedge\text{area}}, \ldots\}$. The calculation can be illustrated by the following table, in which the column of A represents the input, C' and R' represent the subset of C and R, B^* represents the smallest set such that that $B^* = Cn(B^*)$ and $A \cup C'(B^*) \cup R'(B^*) \subseteq B^*$.

A	C'	R'	B^*	$C'(B^*)$	$R'(B^*)$
A4	$\emptyset$	$\emptyset$	$Cn(\{$A4$\})$	$\emptyset$	$\emptyset$
A4	$\{$(A4, size)$\}$	$\{$(size, t25×15)$\}$	$Cn(\{$A4, size, t25×15$\})$	$\{$size$\}$	$\{$t25×15$\}$
A4	C	R	$Cn(\{$A4,size,t25×15, area, ref10$\})$	$\{$size, size $\}$	$\{$t25×15, ref10$\}$

The second row of the table explains the reason of $\top \in \circledcirc_3^+(C, R, \{\text{A4}\})$ and $\top_\top \in \odot_3^+(C, R, \{\text{A4}\})$. The third row explains the reason for t25×15 $\in \circledcirc_3^+(C, R, \{\text{A4}\})$, t25×15$_{\text{size}} \in \odot_3^+(C, R, \{\text{A4}\})$, and the fourth row for t25×15 $\wedge$ ref10.

The following is the proof system of reusable detailed combination. It is an extension of $\mathbf{D}_1^{+'}$, but its formation is simpler in the sense we add one rule called ACTIO but delete both ACTI and ACTO.

Definition 16 (Proof system of reusable detailed combination). *Let $\mathbf{D}_3^+(C, R)$ be the smallest set such that $(\top, \top_\top) \in \mathbf{D}_3^+(C, R)$, $\{(a, x_\top) \mid (a, x) \in R\} \subseteq \mathbf{D}_3^+(C, R)$, $\{(a, \top_p) \mid (a, p) \in C\} \subseteq \mathbf{D}_3^+(C, R)$ and $\mathbf{D}_3^+(C, R)$ is closed under the rules SI, IOEQ, ACTIO*

– *ACTIO: aggregative cumulative transitivity for the intermediate and output: from (a, x_p) and $(a \wedge p \wedge x, y_q)$ to $(a, x \wedge y_{p\wedge q})$.*

and the following rule

– *if $(a, \top_p) \in \mathbf{D}_3^+(C, R)$ and $(a \wedge p, x) \in D(R)$ then $(a, x_p) \in \mathbf{D}_3^+(C, R)$.*

Example 7 (continued). From $C = \{$(A4, size), (t25×15, area)$\}$, $R = \{$(size, t25×15), (area, ref10)$\}$ we can derive expressions (A4,$\top_{\text{size}}$), (t25×15,$\top_{\text{area}}$), (size, t25×15$_\top$) and (area, ref10$_\top$). The following is the derivation:

$$\dfrac{\dfrac{(\text{A4}, \top_{\text{size}}) \qquad \dfrac{(\text{size, t25}\times 15_\top)}{(\text{A4}_{\wedge\top\wedge}\text{size, t25}\times 15_\top)}\,(\text{SI})}{(\text{A4, t25}\times 15\wedge\top_{\text{size}_{\wedge\top}})}\,(\text{ACTIO})}{(\text{A4, t25} \times 15_{\text{size}})}\,(\text{IOEQ})$$

Theorem 7 (Completeness of reusable detailed combination). *Given an arbitrary constitutive normative system C, regulative normative system R and a set A of formulas, $(a, x_p) \in \mathbf{D}_3^+(C, R)$ iff $x_p \in \odot_3^+(C, R, a)$.*

Proof. We prove the special case when A is a singleton $\{a\}$.
(completeness) Here we sketch the proof.
1. Assume $x_p \in \odot_3^+(C, R, a)$. Then there exist $C' \subseteq C, R' \subseteq R$ such that $x_p \in \bigodot_3^+(C', R', a)$
2. By Lemma 4 we have $x \dashv\vdash \bigwedge R(B_a^{C',R'})$ and $p \dashv\vdash \bigwedge C(B_a^{C',R'})$.

3. By clause 5 of Lemma 6 we have $x \dashv\vdash \bigwedge R(B_\omega^{\emptyset,R'})$ and $p \dashv\vdash \bigwedge C(B_\omega^{C',\emptyset})$. Since both C' and R' are finite, by clause 1 of Lemma 6 we know there exist i such that for all $j \geq i$, $x \dashv\vdash \bigwedge R(B_{a\wedge\ldots\wedge x_j}^{\emptyset,R'})$. By clause 2 of Lemma 6 we know there exist i such that for all $j \geq i$, $p \dashv\vdash \bigwedge C(B_{a\wedge\ldots\wedge p_j}^{C',\emptyset})$
4. From 3 we know there exist i such that $p \dashv\vdash \bigwedge C(B_{a\wedge\ldots\wedge p_i}^{C',\emptyset})$, $x \dashv\vdash \bigwedge R(B_{a\wedge\ldots\wedge x_i}^{\emptyset,R'})$, hence $p \dashv\vdash p_{i+1}$, $x \dashv\vdash x_{i+1}$.
5. From 4 and clause 2 of Lemma 5 we know $(a, x_{ip_i}) \in \mathbf{D}_3^+(C', R')$, then use IOEQ we have $(a, x_p) \in \mathbf{D}_3^+(C', R') \subseteq \mathbf{D}_3^+(C, R)$

(soundness) We will prove by induction on the length of derivation that if $(a, x_p) \in \mathbf{D}_3^+(C, R)$ then $x_p \in \odot_3^+(C, R, a)$ There we three base steps:

- (a, x_p) is $(\top, \top_\top)$: Let $C' = \emptyset$ and $R' = \emptyset$. Then for all $B = Cn(B)$, assume $\{\top\} \cup C'(B) \cup R'(B) \subseteq B$. Then we have $R'(B) = \emptyset = C'(B)$. Therefore $\top \dashv\vdash \bigwedge\emptyset \dashv\vdash \bigwedge C'(B)$. $\top \dashv\vdash \bigwedge\emptyset \dashv\vdash \bigwedge R'(B)$. Hence $\top_\top \in \odot_3^+(C, R, \top)$.
- (a, x_p) is $(a, \top_p)$ with $(a, p) \in C$: Let $C' = \{(a, p)\}$, $R' = \emptyset$. Then for all $B = Cn(B)$, assume $\{a\} \cup C'(B) \cup R'(B) \subseteq B$. Then $C'(B) = \{p\}$ and $R'(B) = \emptyset$. Therefore $p \dashv\vdash \bigwedge C'(B)$, $\top \dashv\vdash \bigwedge R'(B)$. Hence $\top_p \in \odot_3^+(C, R, a)$.
- (a, x_p) is $(a, x_\top)$ with $(a, x) \in R$: Let Let $C' = \emptyset, R' = \{(a, x)\}$. Then for all $B = Cn(B)$, assume $\{a\} \cup C'(B) \cup R'(B) \subseteq B$. Then $C'(B) = \emptyset$ and $R'(B) = \{x\}$. Therefore $\top \dashv\vdash \bigwedge C'(B)$, $x \dashv\vdash \bigwedge R'(B)$. Hence $x_\top \in \odot_3^+(C, R, a)$.

Now we move to the inductive steps:

- (a, x_p) is derived by SI:
  1. there exist $(b, x_p) \in \mathbf{D}_3^+(C, R)$ such that $a \vdash b$.
  2. By induction hypothesis, from 1 we know $x_p \in \odot_3^+(C, R, b)$. Therefore there exist $C_1 \subseteq C$ and $R_1 \subseteq R$ such that for all $B_1 = Cn(B)$, if $\{b\} \cup C_1(B) \cup R_1(B) \subseteq B$ then $x \dashv\vdash \bigwedge R_1(B)$ and $p \dashv\vdash \bigwedge C_1(B)$.
  3. Take the C_1 and R_1 of 2. Let $E = Cn(E)$, assume $\{a\} \cup C_1(E) \cup R_1(E) \subseteq E$.
  4. Since $a \vdash b$ and $E = Cn(E)$, from 3 we know $b \in E$. Therefore $\{b\} \cup C_1(E) \cup R_1(E) \subseteq E$. Then from 2 we know $x \dashv\vdash \bigwedge R_1(E)$ and $p \dashv\vdash \bigwedge C_1(E)$.
  5. From 4 we have $x_p \in \odot_3^+(C, R, a)$.
- (a, x_p) is derived by IOEQ:
  1. there exist $(a, y_q) \in \mathbf{D}_3^+(C, R)$ such that $x \dashv\vdash y$ and $p \dashv\vdash q$.
  2. By induction hypothesis, from 1 we know $y_q \in \odot_3^+(C, R, b)$. Therefore there exist $C_1 \subseteq C$ and $R_1 \subseteq R$ such that for all $B_1 = Cn(B)$, if $\{b\} \cup C_1(B) \cup R_1(B) \subseteq B$ then $y \dashv\vdash \bigwedge R_1(B)$ and $q \dashv\vdash \bigwedge C_1(B)$.
  3. From 2 and 1 we know $x \dashv\vdash \bigwedge R_1(B)$ and $p \dashv\vdash \bigwedge C_1(B)$. Therefore $x_p \in \odot_3^+(C, R, a)$.
- (a, x_p) is derived by ACTIO: Assume $(a, x \wedge y_{p\wedge q}) \in \mathbf{D}_3^+(C, R)$ and it is derived by (a, x_p) and $(a \wedge p \wedge x, y_q)$ using the ACTIO rule.
  1. by induction hypothesis we have $x_p \in \odot_3^+(C, R, a)$ and $y_q \in \odot_3^+(C, R, a \wedge p \wedge x)$.
  2. From 1 we know there exist $C_1 \subseteq C$ and $R_1 \subseteq R$ such that for all $B_1 = Cn(B_1)$,

if $\{a\} \cup C_1(B_1) \cup R_1(B_1) \subseteq B_1$ then $x \dashv\vdash \bigwedge R_1(B_1)$ and $p \dashv\vdash \bigwedge C_1(B_1)$. Moreover there exist $C_2 \subseteq C$ and $R_2 \subseteq R$ such that for all $B_2 = Cn(B_2)$, if $\{a \wedge p \wedge x\} \cup C_2(B_2) \cup R_2(B_2) \subseteq B_2$ then $y \dashv\vdash \bigwedge R_2(B_2)$ and $q \dashv\vdash \bigwedge C_2(B_2)$.
3. Let $C_3 = C_1 \cup C_2$ and $R_3 = R_1 \cup R_2$.
4. Let $E_1 = Cn(E_1)$, assume $\{a\} \cup C_3(E_1) \cup R_3(E_1) \subseteq E_1$
5. From 4 and 3 we know $\{a\} \cup C_1(E_1) \cup R_1(E_1) \subseteq E_1$
6. From 5 and 2 we have $x \dashv\vdash \bigwedge R_1(E_1)$ and $p \dashv\vdash \bigwedge C_1(E_1)$.
7. From 6 and 4 we have $x \in E_1$, $p \in E_1$ and further $a \wedge p \wedge x \in E_1$.
8. From 7 and 3 we know $\{a \wedge p \wedge x\} \cup C_2(E_1) \cup R_2(E_1) \subseteq E_1$.
9. From 8 and 2 we know $y \dashv\vdash \bigwedge R_2(E_1)$ and $q \dashv\vdash \bigwedge C_2(E_1)$.
10. From 3, 6 and 9 we know $x \wedge y \dashv\vdash \bigwedge R_3(E_1)$ and $p \wedge q \dashv\vdash \bigwedge C_3(E_1)$.
11. From 3, 4 and 10 we have $x \wedge y_{p \wedge q} \in \odot_3^+(C, R, a)$

Remark: We define a function $f_A^{C,R} : 2^L \to 2^L$ such that $f_A^{C,R}(X) = Cn(A \cup C(X) \cup R(X))$. It can be proved that $f_A^{C,R}$ is monotonic with respect to the set theoretical inclusion $\subseteq$, and $(2^L, \subseteq)$ is a complete lattice. Then by Tarski's fixed point theorem there exist a least fixed point of $f_A^{C,R}$.

Lemma 3. *Let $B_A^{C,R}$ be the least fixed point of the function $f_A^{C,R}$. Then $B_A^{C,R} = \bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$, where $B_{A,0}^{C,R} = Cn(A)$, $B_{A,i+1}^{C,R} = Cn(A \cup C(B_{A,i}^{C,R}) \cup R(B_{A,i}^{C,R}))$.*

Proof. We first prove that $\bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$ is a fixed point of $f_A^{C,R}$. We prove by showing the following:

1. $A \subseteq \bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$: this is because $A \subseteq Cn(A) = B_{A,0}^{C,R} \subseteq \bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$
2. $C(\bigcup_{i=0}^{\infty} B_{A,i}^{C,R}) \subseteq \bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$: For every $x \in C(\bigcup_{i=0}^{\infty} B_{A,i}^{C,R})$, there exist k such that $x \in C(B_{A,k}^{C,R}) \subseteq B_{A,k+1}^{C,R} \subseteq \bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$.
3. $R(\bigcup_{i=0}^{\infty} B_{A,i}^{C,R}) \subseteq \bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$: this can be proved similarly to the above clause.
4. $Cn(\bigcup_{i=0}^{\infty} B_{A,i}^{C,R}) = \bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$: the right-to-left direction is obvious; for the other direction: assume $x \in \bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$, then there exist $x_1, \ldots x_n \in \bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$. Therefore there exist k such that $x_1, \ldots x_n \in B_{A,k}^{C,R}$. Hence $x \in B_{A,k+1}^{C,R} \subseteq \bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$.

With the above four clauses in hand, we can prove that $f_A^{C,R}(\bigcup_{i=0}^{\infty} B_{A,i}^{C,R}) \subseteq \bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$. For the other direction, we prove by induction on i that for every i, $B_{A,i}^{C,R} \subseteq f_A^{C,R}(\bigcup_{i=0}^{\infty} B_{A,i}^{C,R})$. Here we omit the details.

So we have proved that $\bigcup_{i=0}^{\infty} B_{A,i}^{C,R}$ is a fixed point of $f_A^{C,R}$. To prove that it is the least fixed point, we can again prove by induction that for every i, $B_{A,i}^{C,R} \subseteq f_A^{C,R}(B)$, where B is a fixed point of $f_A^{C,R}$. Here we omit the details.

Lemma 4. *Let* $x_p \in \bigodot_3^+(C, R, A)$ *iff* $\forall B = Cn(B)$: *if* $A \cup C(B) \cup R(B) \subseteq B$ *then* $p \dashv\vdash \bigwedge C(B), x \dashv\vdash \bigwedge R(B)$. *Let* $x_p \in \odot_3^+(R, A)$ *iff* $\forall B = Cn(B)$: *if* $A \cup R(B) \subseteq B$ *then* $x \dashv\vdash \bigwedge R(B)$.

(1) If $x_p \in \bigodot_3^+(C, R, A)$*, then* $x \dashv\vdash \bigwedge R(B_A^{C,R})$ *and* $p \dashv\vdash \bigwedge C(B_A^{C,R})$.
(2) If $x \in \odot(R, A)$*, then* $x \dashv\vdash \bigwedge R(B_A^{\emptyset,R}) \dashv\vdash \bigwedge R(B_A^{R,\emptyset})$.

Proof. Assume $x_p \in \bigodot_3^+(C, R, A)$. From Lemma 3, we know that $A \subseteq B_A^{C,R}$. From Lemma 3 and compactness theorem of propositional logic, it can be proved that $Cn(B_A^{C,R}) = B_A^{C,R}$. To prove $C(B_A^{C,R}) \subseteq B_A^{C,R}$, assume $x \in C(B_A^{C,R})$, then there exist $a \in B_A^{C,R}$ such that $(a, x) \in C$. Then by Lemma 3 we know $a \in B_{A,i}^{C,R}$ for some i. Hence $x \in C(B_{A,i}^{C,R}) \subseteq B_{A,i+1}^{C,R} \subseteq B_A^{C,R}$. Similarly we can prove $R(B_A^{C,R}) \subseteq B_A^{C,R}$. Therefore we have $Cn(B_A^{C,R}) = B_A^{C,R}$, $A \cup C(B_{A,i}^{C,R}) \cup R(B_{A,i}^{C,R}) \subseteq B_{A,i}^{C,R}$. Now from $x_p \in \bigodot_3^+(C, R, A)$ we know $x \dashv\vdash \bigwedge R(B_A^{C,R})$ and $p \dashv\vdash \bigwedge C(B_A^{C,R})$.
The case for $x \in \odot(R, A)$ is similar.

Lemma 5. *Given two set of norms* C, R *and one formula* a*, we inductively define the following:*

- $p_0 \dashv\vdash \bigwedge C(B_a^{C,\emptyset})$
- $x_0 \dashv\vdash \bigwedge R(B_{a \wedge p_0}^{\emptyset,R})$
- $p_{i+1} \dashv\vdash \bigwedge C(B_{a \wedge p_0 \wedge x_0 \wedge \ldots \wedge x_i}^{C,\emptyset})$
- $x_{i+1} \dashv\vdash \bigwedge R(B_{a \wedge p_0 \wedge x_0 \wedge \ldots \wedge p+i}^{\emptyset,R})$

Then we have the following: for all i, j,

(1) if $i > j$*, then* $p_i \vdash p_j$ *and* $x_i \vdash x_j$
(2) $(a, x_{ip_i}) \in \mathbf{D}_3^+(C, R)$

Proof. (1) The proof is routine, here we omit it.
(2) We prove by induction on i.

- Base step: Let $i = 0$. By clause (2) of Lemma 4, we know $p_0 \in \odot(C, a)$ and $x_0 \in \bigcirc(R, a \wedge p_0)$. Therefore by the completeness so system $\bigcirc$ we know $(a, p_0) \in D(C)$ and $(a \wedge p_0, x_0) \in D(R)$. Then by we know $(a, x_{0p_0}) \in \mathbf{D}_1^+(C, R) \subseteq \mathbf{D}_3^+(C, R)$.
- Inductive step: Assume $(a, x_{kp_k}) \in \mathbf{D}_3^+(C, R)$. By clause (1) of this lemma we know $a \wedge x_k \wedge p_k \dashv\vdash a \wedge x_0 \wedge p_0 \wedge \ldots \wedge x_k \wedge p_k$. By the definition of p_{k+1} and clause (2) of Lemma 4 we know $p_{k+1} \in \bigcirc(C, a \wedge x_0 \wedge p_0 \wedge \ldots \wedge x_k \wedge p_k)$, and similarly we have $x_{k+1} \in \bigcirc(R, a \wedge x_0 \wedge p_0 \wedge \ldots \wedge x_k \wedge p_k \wedge p_{k+1})$. Therefore we have $(a \wedge x_0 \wedge p_0 \wedge \ldots \wedge x_k \wedge p_k, p_{k+1}) \in D(C)$, $(a \wedge x_0 \wedge p_0 \wedge \ldots \wedge x_k \wedge p_k \wedge p_{k+1}, x_{k+1}) \in D(R)$. Hence we know $(a \wedge x_0 \wedge p_0 \wedge \ldots \wedge x_k \wedge p_k, x_{k+1p_{k+1}}) \in \mathbf{D}_1^+(C, R) \subseteq \mathbf{D}_3^+(C, R)$. Then apply SI we $(a \wedge x_k \wedge p_k, x_{k+1p_{k+1}}) \in \mathbf{D}_3^+(C, R)$. Note that $(a, x_{kp_k}) \in \mathbf{D}_3^+(C, R)$, hence by ACTIO we have $(a, x_{k+1p_{k+1}}) \in \mathbf{D}_3^+(C, R)$.

Lemma 6. *Given two set of norms* C, R *and one formula* a*, we define the following:*

- $B_{\omega}^{C,\emptyset} = B_a^{C,\emptyset} \cup \bigcup_{i=0}^{\infty} B_{a\wedge p_0\wedge x_0\wedge\ldots\wedge x_i}^{C,\emptyset}$
- $B_{\omega}^{\emptyset,R} = \bigcup_{i=0}^{\infty} B_{a\wedge p_0\wedge x_0\wedge\ldots\wedge p_i}^{\emptyset,R}$

Then we have the following: for all i,

(1) for all i, $B_{a\wedge p_0\wedge x_0\wedge\ldots\wedge p_i}^{\emptyset,R} \subseteq B_{a\wedge p_0\wedge x_0\wedge\ldots\wedge p_{i+1}}^{\emptyset,R}$
(2) for all i, $B_{a\wedge p_0\wedge x_0\wedge\ldots\wedge x_i}^{C,\emptyset} \subseteq B_{a\wedge p_0\wedge x_0\wedge\ldots\wedge x_{i+1}}^{C,\emptyset}$
(3) for all i, $B_{a\wedge p_0\wedge x_0\wedge\ldots\wedge p_i}^{\emptyset,R} \subseteq B_{a\wedge p_0\wedge x_0\wedge\ldots\wedge x_i}^{C,\emptyset}$
(4) for all i, $B_{a\wedge p_0\wedge x_0\wedge\ldots\wedge x_i}^{C,\emptyset} \subseteq B_{a\wedge p_0\wedge x_0\wedge\ldots\wedge p_{i+1}}^{\emptyset,R}$
(5) $B_{\omega}^{C,\emptyset} = B_{\omega}^{\emptyset,R} = B_a^{C,R}$

Proof. The first four clauses can be proved routinely by careful check. Here we omit it. With the first four clauses in hand, the fifth clause can be derived easily.

At a first glance it seems that such a reusable approach conflates the distinction between "p is the case" and "p is obligatory." This is true in the sense that both facts and detached obligations are used as input to detach new obligations. Of course "p is the case" and "p is obligatory" are semantically different, but we believe such difference is not a sufficient reason to reject aggregative deontic detachment. For defense of aggregative deontic detachment the readers are suggested to consult to Parent and van der Torre [19].

6 Related work

Grossi and Jones [8] track the distinction between constitutive norms and regulative norms to at least Rawls [20]. Searle [22] uses 'X counts as Y in context C" as a canonical presentation of constitutive norms. Jones and Sergot [12] formalize the context as an institution. Grossi et al. [9] formalize the context as a set norms.

Grossi and Jones [8] present a classification of approaches to formalize constitutive norms, and we refer to their chapter for further background on the logic of constitutive norms. On one hand all logics of constitutive norms are orthogonal to our work because we focus on the combination of constitutive and regulative norms. On the other hand, we adopt aggregative input/output logic as our logic for constitutive norms, which is different from all the approaches summarized by Grossi and Jones [8].

Modalities can be combined using possible world semantics. Boutilier [3] introduces a model $M = (W, \geq_1, \geq_2, V)$ with $\geq_1, \geq_2$ total pre-orders over W, reflecting normality and preference respectively. Each pre-order defines a classical Danielsson-Hansson-Lewis dyadic operator $\bigcirc^i$. Roughly, Boutilier defines a modality $\bigcirc^{12}(B|A)$ as the best of the most normal A worlds satisfy B:

- $M, w \models \bigcirc^i(B \mid A)$ iff $opt_{\geq_i}(\|A\|) \subseteq \|B\|$, for $i = 1, 2$
- $M, w \models \bigcirc^{12}(B \mid A)$ iff $opt_{\geq_2}(opt_{\geq_1}(\|A\|)) \subseteq \|B\|$

Here $opt_{\geq_i}(S) = \{w \in S | \forall u \in S, w \geq_i u\}$, for every $S \subseteq W$. According to the semantics, the combined modality does not satisfy the combination properties discussed in this paper:

Observation 1 $\{\bigcirc^1(q \mid p), \bigcirc^2(r \mid q)\} \not\models \bigcirc^{12}(r \mid p)$

Lang and van der Torre [13] define $\bigcirc^{12}(B \mid A)$ in the same models by: the most normal $A \wedge B$ is preferred to the most normal $A \wedge \neg B$, and they compare their definition with Boutilier's. Observation 1 also hold for Lang and van der Torre's combination. A further comparison between these modal logic approaches and our approach is left for further research.

7 Future research

The reusable combination is formed by adding ACT to throughput combination. We can form a weaker version of reusable combination by adding ACT to basic combination.

Makinson and van der Torre [17] developed input/output logic not only for obligations:

> "In a range of contexts, one comes across processes resembling inference, but where input propositions are not in general included among outputs, and the operation is not in any way reversible. Examples arise in contexts of conditional obligations, goals, ideals, preferences, actions, and beliefs. Our purpose is to develop a theory of such input/output operations."

Therefore, in future research we want to investigate whether our framework can be used also for combining other modalities. Consider the well known problem of combining beliefs and desires: you may desire to go to the dentist, you may believe that going to the dentist means that you will have pain, but you do not desire to have pain.

We can model this in our framework, if C stands for beliefs (or knowledge) and R are desires (or obligations), then we have $C = (dentist, pain)$ and $R = (\top, dentist)$. In all the three abstract combinations we have *dentist* $\in \bigcirc_{*s}(C, R, \emptyset)$ but not *pain* $\in \bigcirc_{*s}(C, R, \emptyset)$.

8 Summary

To reason with constitutive and regulative norms, one has to choose a logic for the constitutive norms, a logic for the regulative norms, and a semantics to combine these two logics. In this paper we consider the question which semantics to choose for combining constitutive and regulative norms, a topic which has not raised much attention thus far, without committing ourselves to particular logics for constitutive or regulative norms. To make our analysis general, we use the 'minimal' logic introduced by Parent and van der Torre. Nevertheless, it contains two assumptions. First, strengthening of the input seems to reflect that rules do not have exceptions, whereas both constitutive and regulative norms encountered in practice often do have such exceptions. We do not consider this a limitation, because if we add priorities or a normality relate to reflect prima facie norms, exactly the same analysis can be given. Second, Parent and van der Torre's logic satisfies aggregative deontic detachment. This is only a weak notion of deontic detachment, and some kind of deontic detachment is needed in the logics to be able to define the reusability semantics. We thus distinguished three semantics to combine constitutive and regulative norms:

The simple-minded semantics is the least committed, and thus the safest one to use. It clearly distinguishes the input, intermediate facts and output obligations, there is no possible source for confusion. It may be used, for example, when the input may not be true at the intermediate stage. For example, in legal interpretation the input may contains a bicycle which is also a vehicle, but the bicycle may not count as a vehicle in the legal sense. Using the basic semantics, the intermediate facts may not contain a fact that the bicycle is a vehicle, whereas the input does. The proof system shows that the extension of the semantic is minimal, in the sense that it contains only a transitivity proof rule for the combination (Definition 4 and Theorem 2).

The throughput semantics includes the input among the intermediate facts. The proof system shows that the difference with the simple-minded semantics is small, we just have to replace the transitivity axiom by a cumulative transitivity rule (Theorem 4). However, it means that for example in the bicycle case, we need to introduce another concept in the intermediate facts to represent that the bicycle does not count as a vehicle in the legal sense. In many common examples, it seems that the throughput semantics is preferred to the basic semantics.

The reusability semantics considers as the intermediate state the facts closed under both the constitutive and regulative norms. This seems very strong, but the proof system shows that this corresponds precisely to the aggregative cumulative transitivity rule for the combined system (Theorem 6). So if this rule is desired, then this semantics has to be chosen. For example, if we start from a system satisfying ACT, and then refining it with systems for constitutive and regulative norms, then we need to refine it in this way.

In this paper we introduce new logics for combining constitutive and regulative norms, deriving expressions x_p for x is obligatory because of the intermediate concepts p, or simply x meaning x is obligatory without referring to intermediate concept. We have extended each of the three above systems with a proof system for these refined expressions.

We generalized input/output logic by considering two sets of norms. It can be further extended to LIONS, as foreseen by Makinson and van der Torre. And to refer tot he special topic of DEON14, it may be a first step towards a Kratzer style semantics of natural language, because Kratzer's semantics also combines various kinds of ordering bases.

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