Jury Theorem under Uncertainty

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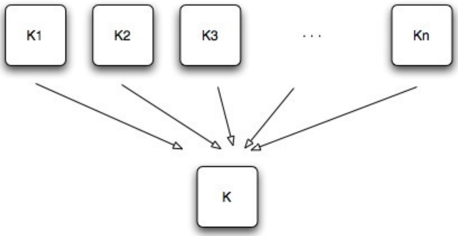
Merging

- Contradictory beliefs/goals coming from different sources
- Propositional Logic
- no priority (same reliability, hierarchical importance, ...)



- Base K = a set of propositional formulae
- Profile $E = \{K_1, \ldots, K_n\}$
- Integrity Constraints = a propositional formula μ
- Merging operator $\triangle : E, \mu \longrightarrow K$

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	Ideal Process	Practical Process

Definitions

- A base φ is a (finite set of) propositional formula
- A profile E is a multi-set of bases $E = \{\varphi_1, \ldots, \varphi_n\}$
- $\bigwedge E$ denotes the conjunction of the bases of E, i.e. $\bigwedge E = \varphi_1 \wedge \ldots \wedge \varphi_n$
- A profile E is consistent if and only if $\bigwedge E$ is consistent
  We will note $Mod(E)$ the models of $\bigwedge E$
- Equivalence between profiles :
  - Let E_1, E_2 be two profiles. E_1 and E_2 are *equivalent*, noted $E_1 \equiv E_2$, iff there exists a bijection f from $E_1 = \{\varphi_1^1, \ldots, \varphi_n^1\}$ to $E_2 = \{\varphi_1^2, \ldots, \varphi_n^2\}$ such that $\vdash f(\varphi) \leftrightarrow \varphi$.

Belief Merging vs. Goal Merging

- Logical properties for merging
- Same properties for belief merging and goal merging
- Is it possible to discriminate these two tasks?

Logical Properties

$\triangle$ is a merging with integrity constraints operator (IC merging operator) if it satisfies the following properties :

(IC0) $\triangle_\mu(E) \vdash \mu$

(IC1) If μ is consistent, then $\triangle_\mu(E)$ is consistent

(IC2) If $\bigwedge E$ is consistent with μ, then $\triangle_\mu(E) = \bigwedge E \wedge \mu$

(IC3) If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\triangle_{\mu_1}(E_1) \equiv \triangle_{\mu_2}(E_2)$

(IC4) If $\varphi \vdash \mu$ and $\varphi' \vdash \mu$, then $\triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi \nvdash \perp \Rightarrow \triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi' \nvdash \perp$

(IC5) $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2) \vdash \triangle_\mu(E_1 \sqcup E_2)$

(IC6) If $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$ is consistent, then $\triangle_\mu(E_1 \sqcup E_2) \vdash \triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$

(IC7) $\triangle_{\mu_1}(E) \wedge \mu_2 \vdash \triangle_{\mu_1 \wedge \mu_2}(E)$

(IC8) If $\triangle_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(E) \vdash \triangle_{\mu_1}(E)$

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  - Condorcet's Jury Theorem [Condorcet 1785]
- Merging?

Outline

- Condorcet's Jury Theorem
- Jury Theorem under Uncertainty
- Truth Tracking Postulate
- Some Experiments on Convergence Speed
- Conclusion

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- Suppose
  - 2 alternatives ω and $\omega^\star$
  - $\omega^\star$ is the correct answer
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    - independent
    - reliable (more than half-a-chance to give the correct answer)
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Consider a real number $p^\star \in [0, 1[$ and a profile E from a set of n independent agents who have the same reliability $p > p^\star$. The probability that the score of the correct answer exceeds $np^\star$ tends to 1 when n tends to infinity.

$$P(s_a(\omega^\star) > np^\star) \xrightarrow[n \to \infty]{} 1$$

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- If all individuals share the same reliability $p > \kappa$, then the correct answer belongs to the set of states returned by the κ-quota rule in the limit.
- Problem: The rule which always returns the set of all alternatives $\{\omega^\star, \omega_1, \ldots, \omega_{k-1}\}$ achieves the same result !

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- Improved reliability: an individual is R4-reliable if it is more reliable than incompetent

Jury Theorem under Uncertainty III

Theorem

Let $\{\omega^{\star}, \omega_1, \ldots, \omega_{k-1}\}$ *be a set of possible worlds and let E be a profile from a set of n independent, homogenous and R4-reliable individuals. Then the probability than the correct answer is identified (i.e., is the only chosen alternative) by the majority tends to 1 as the group size increases, i.e.,* $\forall i \in \{1, \ldots, k-1\}$,

$$P(s_a(\omega^{\star}) > s_a(\omega_i)) \xrightarrow[n \to \infty]{} 1$$

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- R4-reliability extends reliability in [List Goodin 2001] and [Condorcet 1785]
- Jury Theorem under Uncertainty extends [List Goodin 2001] Theorem and Condorcet's Jury Theorem

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- R4-reliability extends reliability in [List Goodin 2001] and [Condorcet 1785]
- Jury Theorem under Uncertainty extends [List Goodin 2001] Theorem and Condorcet's Jury Theorem
- The majority method in this Jury Theorem under Uncertainty is approval voting. Thus this theorem shows that approval voting is the appropriate truth-tracking method for voting on k ($k > 2$) alternatives

Distance-based merging operators

- Let d be a distance between interpretations and f be an aggregation function. The merging operator $\triangle^{d,f}(E)$ is defined by:

$$mod(\triangle_{\mu}^{d,f}(E)) = \min(mod(\mu), \leq_E)$$

  where the pre-order $\leq_E$ on $\mathcal{W}$ induced by E is defined by:
  - $\omega \leq_E \omega'$ if and only if $d(\omega, E) \leq d(\omega', E)$, where
  - $d(\omega, E) = f_{K \in E}(d(\omega, K))$, where
  - $d(\omega, K) = min_{\omega' \models K} d(\omega, \omega')$
- Examples of distances:
  - drastic distance d_D
  - Hamming (Dalal) distance d_H
- Examples of aggregation functions:
  - sum (Σ)
  - leximax ($Gmax$)
  - leximin ($Gmin$)

Truth Tracking Postulate for Merging Operators

Let $\triangle$ be a merging operator

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Proposition

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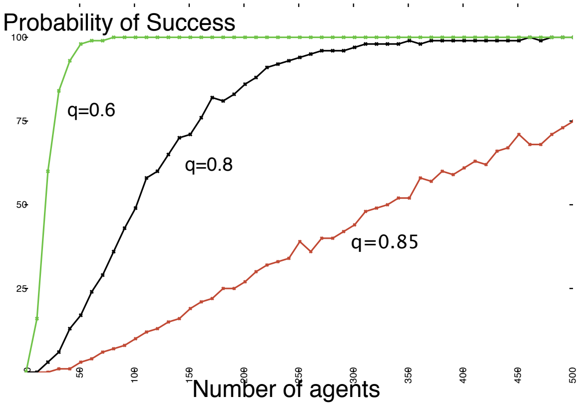
- $\triangle^{d_H,\Sigma}$ *does not satisfy* **(TT)**
- $\triangle^{d_D,\Sigma}$ *satisfies* **(TT)**
- *For any pseudo-distance d, $\triangle^{d,Gmin}$ satisfies* **(TT)**

Some Experimental Results: Convergence Speed (7 variables, p=0.9)

- p: agents reliability ($p = P(\omega^\star \in K_i)$)
- q: agents incompetence ($q = P(\omega \neq \omega^\star \in K_i)$)



Some Experimental Results: Convergence Speed (7 variables, p=0.3)

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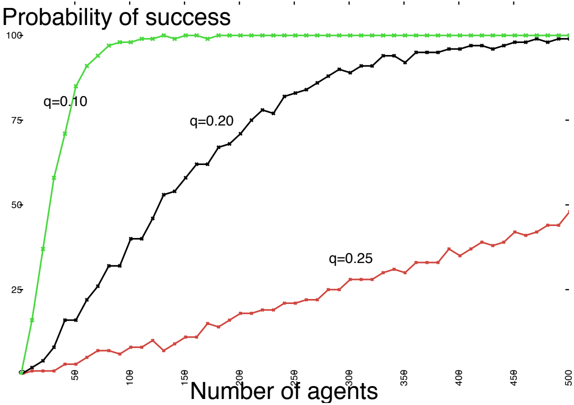

Conclusion, Related Work and Perspectives

- Conclusion
  - Jury Theorem under Uncertainty
  - Difference between belief merging and goal merging
  - Synthesis view versus epistemic view of merging
  - Truth tracking postulate
- Related Work
  - Truth Tracking for Judgement Aggregation
    [BOVENS RABINOWICZ 2006] [PIGOZZI HARTMANN 2007]
- Perspectives
  - Releasing assumptions (homogeneity, etc...)
    - [OWEN-GROFMAN-FELD 1989]:
      The average reliability is greater than 0.5
  - Judgment Aggregation methods and Maximum likelihood