# Contextual Deontic Logic

Leendert W.N. van der Torre
EURIDIS and Tinbergen Institute
torre@cs.few.eur.nl

Yao-Hua Tan
EURIDIS
ytan@euridis.fbk.eur.nl

Erasmus University Rotterdam
P.O. Box 1738
3000 DR Rotterdam
The Netherlands
tel (+31)10-4082601 fax (+31)10-4526134

**Abstract**

In this article we propose contextual deontic logic (CDL). Contextual obligations are written as $O(\alpha|\beta\backslash\gamma)$, and are to be read as '$\alpha$ should be the case if $\beta$ is the case, unless $\gamma$ is the case'. The unless clause is analogous to the justification in Reiter's default rules. We show how contextual obligations can be used to solve certain aspects of contrary-to-duty paradoxes of dyadic deontic logic.

## 1 Contrary-to-duty reasoning

In recent years several researchers have argued that deontic logic is a useful tool to model reasoning in (legal) knowledge-based systems [JS92, RL92, Smi94, Roy96]. The problem, however, is that deontic logic is hampered by the so-called deontic paradoxes. The contrary-to-duty paradoxes like the notorious Chisholm paradox are the classic benchmark problems of deontic logics, which have initiated developments of monadic deontic logics [Chi63, For84], dyadic deontic logics [Tom81] and temporal deontic logics [vE82]. In this article we analyze certain aspects of the paradoxes in dyadic deontic logics, in which an obligation $O(\alpha|\beta)$ is read as '$\alpha$ should be the case if $\beta$ is the case.' An obligation $O(\alpha|\beta)$ is a *contrary-to-duty* obligation of the *primary* obligation $O(\alpha_1|\beta_1)$ if and only if $\beta \wedge \alpha_1$ is inconsistent, as represented in Figure 1.

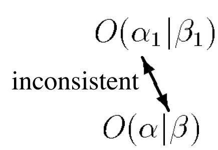


Figure 1: $O(\alpha|\beta)$ is a contrary-to-duty obligation of $O(\alpha_1|\beta_1)$

The following example illustrates that the derivation of the obligation $O(\alpha_1|\neg\alpha_2)$ from the obligation $O(\alpha_1 \wedge \alpha_2|\top)$ is a fundamental problem underlying several contrary-to-duty paradoxes. Hence, the underlying problem of the contrary-to-duty paradoxes is that a contrary-to-duty obligation can be derived from its primary obligation.

**Example 1 (contrary-to-duty paradoxes)** *Assume a dyadic deontic logic that validates at least substitution of logical equivalents and the following inference patterns* Restricted Strengthening of the

Antecedent *(*RSA*)*, Weakening of the Consequent *(*WC*)*, Conjunction *(*AND*) and a version of* Deontic Detachment *(*DD$'$*), in which* $\overset{\leftrightarrow}{\Diamond}$ *is a modal operator (that will be explained later) and* $\overset{\leftrightarrow}{\Diamond}\,\phi$ *is true for all consistent propositional formulas* $\phi$*.*

$$\text{RSA}: \frac{O(\alpha|\beta_1), \overset{\leftrightarrow}{\Diamond}(\alpha \wedge \beta_1 \wedge \beta_2)}{O(\alpha|\beta_1 \wedge \beta_2)} \qquad \text{WC}: \frac{O(\alpha_1|\beta)}{O(\alpha_1 \vee \alpha_2|\beta)}$$

$$\text{AND}: \frac{O(\alpha_1|\beta), O(\alpha_2|\beta)}{O(\alpha_1 \wedge \alpha_2|\beta)} \qquad \text{DD}': \frac{O(\alpha|\beta), O(\beta|\gamma)}{O(\alpha \wedge \beta|\gamma)}$$

*Furthermore, consider the sets* $S = \{O(\neg k|\top), O(g \wedge k|k)\}$*,* $S' = \{O(a|\top), O(t|a), O(\neg t|\neg a)\}$*, and* $S'' = \{O(\neg a \mid \top), O(a \vee p \mid \top), O(\neg p \mid a)\}$*, where* $\top$ *stands for any tautology.* $S$ *formalizes the Forrester paradox [For84] when* $k$ *is read as 'killing someone' and* $g \wedge k$ *as 'killing someone gently,'* $S'$ *formalizes the Chisholm paradox [Chi63] when* $a$ *is read as 'a certain man going to the assistance of his neighbors' and* $t$ *as 'the man telling his neighbors that he will come,' and finally,* $S''$ *formalizes the apples-and-pears example [TvdT96] when* $a$ *is read as 'buying apples' and* $p$ *as 'buying pears.' The last obligation of each premise set is a contrary-to-duty obligation of the first obligation of the set, because its antecedent is contradictory with the consequent of the latter. The paradoxical consequences of the sets of obligations are represented in Figure 2. The underlying problem of the counterintuitive derivations is the derivation of the obligation* $O(\alpha_1|\neg\alpha_2)$ *from* $O(\alpha_1 \wedge \alpha_2|\top)$ *by* WC *and* RSA*: respectively the derivation of* $O(\neg(g \wedge k)|k)$ *from* $O(\neg k|\top)$*,* $O(t|\neg a)$ *from* $O(a \wedge t|\top)$*, and* $O(p|a)$ *from* $O(\neg a \wedge p|\top)$*.*

$$\dfrac{\dfrac{\dfrac{O(\neg k|\top)}{O(\neg(g \wedge k)|\top)}\,\text{WC}}{O(\neg(g \wedge k)|k)}\,\text{RSA} \qquad O(g \wedge k|k)}{O(\neg(g \wedge k) \wedge (g \wedge k)|k)}\,\text{AND}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{O(t|a) \quad O(a|\top)}{O(a \wedge t|\top)}\,\text{DD}'}{O(t|\top)}\,\text{WC}}{O(t|\neg a)}\,\text{RSA} \qquad O(\neg t|\neg a)}{O(t \wedge \neg t|\neg a)}\,\text{AND} \qquad \dfrac{\dfrac{\dfrac{\dfrac{O(\neg a|\top) \quad O(a \vee p|\top)}{O(\neg a \wedge p|\top)}\,\text{AND}}{O(p|\top)}\,\text{WC}}{O(p|a)}\,\text{RSA} \qquad O(\neg p|a)}{O(p \wedge \neg p|a)}\,\text{AND}$$

Figure 2: Three contrary-to-duty paradoxes

There are two types of dyadic deontic logics, dependent on how the antecedent is interpreted. The first type, as advocated by Chellas [Che74, Alc93], defines a dyadic obligation in terms of a monadic obligation by $O(\alpha \mid \beta) =_{def} \beta > O\alpha$, where '$>$' is a strict implication. These dyadic deontic logics have strengthening of the antecedent, but they cannot represent the contrary-to-duty paradoxes in a consistent way. Dyadic deontic logics of the second type, as introduced by Hansson [Han71] and further investigated by Lewis [Lew74], do not have strengthening of the antecedent and therefore they can represent the paradoxes. Intuitively, the solution of these logics is that the antecedent of the dyadic obligations is interpreted as a kind of 'context'. For example, in the Forrester paradox the derivation of the obligation $O(\neg(g \wedge k)|k)$ from $O(\neg k|\top)$ is counterintuitive, because in the context where you kill, it is not obligatory not to kill gently (whereas this is obligatory in the most general context). Because there are many different problems related to the Forrester and Chisholm paradoxes, we restrict our analysis to the apples-and-pears example. In the contextual interpretation of the apples-and-pears example, the derivation of the obligation $O(p \mid a)$ from $O(\neg a \mid \top)$ and $O(a \vee p \mid \top)$ is counterintuitive, because in the context where apples are bought, it is not obligatory to buy pears (whereas this is obligatory in the most general context).

In this paper, we propose a solution for the paradoxes based on contextual obligations. A contextual obligation, written as $O(\alpha \,|\, \beta \backslash \gamma)$, is an extension of a dyadic obligation $O(\alpha \,|\, \beta)$ with an unless clause $\gamma$. The unless clause can be compared to the justification in a Reiter default '$\alpha$ is normally the case if $\beta$ is the case unless $\gamma$ is the case,' written as $\beta : \neg\gamma/\alpha$ [Rei80]. For example, 'birds fly unless they are penguins' can be represented by $b : \neg p/f$, and 'penguins do not fly' by $(b \wedge p) : \top/\neg f$. Hence, the unless clause is analogous to the justification of a Reiter default, which means that it formalizes a kind of consistency check. Contextual deontic logic has in contrast to Reiter's default logic intuitive preference-based semantics.

This paper is organized as follows. In Section 2 we give the solution of the apples-and-pears problem in labeled deontic logic LDL. In Section 3 we introduce contextual obligations $O(\alpha \,|\, \beta \backslash \gamma)$, and we show how they solve the apples-and-pears problem. Finally, in Section 4 we mention some interesting connections with logics of defeasible reasoning and qualitative decision theory.

# 2 Labeled obligations

In [vdTT95] we introduced labeled deontic logic LDL, a logic inspired by contextual logic [BT96]. Labeled obligations $O(\alpha|\beta)_L$ can *roughly* be read as '$\alpha$ ought to be the case, if $\beta$ is the case, because of $L$.'

## 2.1 Implicit and explicit obligations

To illustrate the distinction between implicit and explicit obligations, we recall the well-known distinction between implicit and explicit knowledge. The latter distinction originates in the logical omniscience problem: in principle, an agent cannot know all logical consequences of his knowledge. The benchmark example is that knowledge of the laws of mathematics does not imply knowledge of the theorem of Fermat. That is, an agent does not explicitly know the theorem of Fermat, she only implicitly knows it. Analogously, explicit obligations are not deductively closed, in contrast to implicit obligations.

Several researchers make a distinction between imperatives and obligations, although many researchers hold them as essentially the same. Explicit obligation can be used to formalize imperatives, and implicit obligations can be used to formalize the 'usual' type of obligations. The idea behind labeled obligations is to represent the explicit obligation, of which the implicit obligation is derived, in the label. The label is the reason for the obligation. If we make the distinction between imperatives and obligations, then the label $L$ of the obligation $O(\alpha \mid \beta)_L$ represents the imperatives from which the obligation is derived. This explains our reading of the label obligation $O(\alpha|\beta)_L$: '$\alpha$ ought to be the case if $\beta$ is the case, because of the imperatives $L$.'

We can use labeled deontic logic to solve the contrary-to-duty paradoxes, because we use the label to check that a derived obligation is not a contrary-to-duty obligation of its premises. Remember that we can test for CTD with a consistency check, see Figure 1. The label of an obligation represents the consequents of the premises from which the obligation is derived. In labeled deontic logic we use a consistency check of the label of the obligation with its antecedent. If the label and the antecedent are consistent, then the derived obligation is not a contrary-to-duty of its premises.

## 2.2 Labeled obligations

In this section we introduce a deontic version of a labeled deductive system as it was introduced by Gabbay in [Gab91]. The language of dyadic deontic logic is enriched by allowing labels in the dyadic obligations. Roughly speaking, the label $L$ is a record of the consequents of all the premises that are used in the derivation of $O(\alpha|\beta)$.

**Definition 1 (language of** LDL**)** *The language of labeled deontic logic is a propositional base logic $\mathcal{L}$ and labeled dyadic conditional obligations $O(\alpha|\beta)_L$, with $\alpha$ and $\beta$ sentences of $\mathcal{L}$, and $L$ a set of sentences of $\mathcal{L}$.*

Each labeled obligation occuring as a premise has its own consequent in its label. This represents that the premises are explicit obligations, because it is derived 'from itself.'

**Definition 2 (premises of** LDL**)** *A labeled obligation which has its own consequent as its label is called a premise.*

We assume that the antecedent and the label of an obligation are always consistent. The label of an obligation derived by an inference rule is the union of the labels of the premises used in this inference rule. Below are some labeled versions of inference schemes. We write $\overset{\leftrightarrow}{\Diamond} L$ for a consistency check of a set of formulas.

$$\text{RSA}_V : \frac{O(\alpha \mid \beta_1)_L, \overset{\leftrightarrow}{\Diamond}(L \cup \{\beta_1 \wedge \beta_2\})}{O(\alpha \mid \beta_1 \wedge \beta_2)_L}$$

$$\text{WC}_V : \frac{O(\alpha_1 \mid \beta)_L}{O(\alpha_1 \vee \alpha_2 \mid \beta)_L}$$

$$\text{RDD}'_V : \frac{O(\alpha|\beta)_{L_1}, O(\beta|\gamma)_{L_2}, \overset{\leftrightarrow}{\Diamond}(L_1 \cup L_2 \cup \{\gamma\})}{O(\alpha \wedge \beta \mid \gamma)_{L_1 \cup L_2}}$$

$$\text{RAND}_V : \frac{O(\alpha_1 \mid \beta)_{L_1}, O(\alpha_2 \mid \beta)_{L_2}, \overset{\leftrightarrow}{\Diamond}(L_1 \cup L_2 \cup \{\beta\})}{O(\alpha_1 \wedge \alpha_2 \mid \beta)_{L_1 \cup L_2}}$$

Informally, the premises used in the derivation tree are not violated by the antecedent of the derived obligation, or, alternatively, the derived obligation is not a contrary-to-duty obligation of these premises. We say that the labels formalize the assumptions on which an obligation is derived, and the consistency check $\overset{\leftrightarrow}{\Diamond}$ checks whether the assumptions are violated. The following example illustrates that the labeled deductive system gives the intuitive reading of the Apples-and-Pears example.

**Example 2 (Apples-and-Pears, continued)** *Assume a labeled deductive system that validates at least the inference patterns* $\text{RSA}_V$*,* $\text{RAND}_V$ *and* $\text{WC}_V$*. Consider the premise set of labeled obligations $S = \{O(a \vee p|\top)_{a \vee p}, O(\neg a|\top)_{\neg a}\}$ as premise, where $a$ can be read as 'buying apples' and $p$ as 'buying pears'. In Figure 3 below it is shown how the derivation in Figure 2 is blocked.*

The apples-and-pears example in labeled deontic logic showed an important property of dyadic deontic logics with a contextual interpretation of the antecedent, namely that the context is restricted to non-violations of premises. If the antecedent is a violation, i.e. if the derived obligation would be a contrary-to-duty obligation, then the derivation is blocked. Obviously, as a logic the labeled deductive system is quite limited, if only because it lacks a semantics. In the following section, we consider contextual deontic logic, which has an intuitive preference-based semantics.

$$\dfrac{O(a \vee p|\top)_{\{a\vee p\}} \quad O(\neg a|\top)_{\{\neg a\}}}{O(\neg a \wedge p|\top)_{\{a\vee p,\neg a\}}}\ \text{AND}$$
$$- - - - - - - - - - \ (\text{SA/RSA})$$
$$\dfrac{O(\neg a \wedge p|a)_{\{a\vee p,\neg a\}}}{O(p|a)_{\{a\vee p,\neg a\}}}\ \text{WC}$$

$$\dfrac{\dfrac{O(a \vee p|\top)_{\{a\vee p\}} \quad O(\neg a|\top)_{\{\neg a\}}}{O(\neg a \wedge p|\top)_{\{a\vee p,\neg a\}}}\ \text{AND}}{O(p|\top)_{\{a\vee p,\neg a\}}}\ \text{WC}$$
$$- - - - - - - - \ (\text{SA/RSA})$$
$$O(p|a)_{\{a\vee p,\neg a\}}$$

Figure 3: The apples-and-pears example

# 3 Contextual obligations

Contextual obligations are formalized in Boutilier's modal preference logic CT4O, a bimodal propositional logic of inaccessible worlds. For the details and completeness proof of this logic see [Bou94a]. In the logic we abstract from actions, time and individuals.

**Definition 3 (CT4O)** *The logic CT4O is a propositional bimodal system with the two normal modal connectives $\Box$ and $\overleftarrow{\Box}$. Dual 'possibility' connectives $\Diamond$ and $\overleftarrow{\Diamond}$ are defined as usual and two additional modal connectives $\overleftrightarrow{\Box}$ and $\overleftrightarrow{\Diamond}$ are defined as follows.*

$$\Diamond\alpha =_{def} \neg\Box\neg\alpha \qquad \overleftrightarrow{\Box}\alpha =_{def} \Box\alpha \wedge \overleftarrow{\Box}\alpha$$
$$\overleftarrow{\Diamond}\alpha =_{def} \neg\overleftarrow{\Box}\neg\alpha \qquad \overleftrightarrow{\Diamond}\alpha =_{def} \Diamond\alpha \vee \overleftarrow{\Diamond}\alpha$$

*CT4O is axiomatized by the following axioms and inference rules.*

| | | | |
|---|---|---|---|
| **K** | $\Box(\alpha\to\beta)\to(\Box\alpha\to\Box\beta)$ | **Nes** | *From $\alpha$ infer $\overleftrightarrow{\Box}\alpha$* |
| **K′** | $\overleftarrow{\Box}(\alpha\to\beta)\to(\overleftarrow{\Box}\alpha\to\overleftarrow{\Box}\beta)$ | **MP** | *From $\alpha\to\beta$ and $\alpha$ infer $\beta$* |
| **T** | $\Box\alpha\to\alpha$ | | |
| **4** | $\Box\alpha\to\Box\Box\alpha$ | | |
| **H** | $\overleftrightarrow{\Diamond}(\Box\alpha\wedge\overleftarrow{\Box}\beta)\to\overleftrightarrow{\Box}(\alpha\vee\beta)$ | | |

*Kripke models $M = \langle W, \leq, V\rangle$ for CT4O consist of $W$, a set of worlds, $\leq$, a binary transitive and reflexive accessibility relation, and $V$, a valuation of the propositional atoms in the worlds. The partial pre-ordering $\leq$ expresses preferences: $w_1 \leq w_2$ iff $w_1$ is as preferable as $w_2$. The modal connective $\Box$ refers to accessible worlds and the modal connective $\overleftarrow{\Box}$ to inaccessible worlds.*

$$M, w \models \Box\alpha \text{ iff } \forall w' \in W \text{ if } w' \leq w, \text{ then } M, w' \models \alpha$$
$$M, w \models \overleftarrow{\Box}\alpha \text{ iff } \forall w' \in W \text{ if } w' \not\leq w, \text{ then } M, w' \models \alpha$$

Contextual obligations are defined in CT4O as follows. In this paper, we do not discuss the properties of $>_s$ but we focus on the properties of the contextual obligations.[1]

**Definition 4 (CDL)** *The logic CDL is the logic CT4O extended with the following definitions of contextual obligations. The contextual obligation '$\alpha$ should be the case if $\beta$ is the case unless $\gamma$ is the case', written as $O(\alpha|\beta\backslash\gamma)$, is defined as a strong preference of $\alpha\wedge\beta\wedge\neg\gamma$ over $\neg\alpha\wedge\beta$.*

[1] The preference relation $>_s$ is quite weak. For example, it is not anti-symmetric (we cannot derive $\neg(\alpha_2 >_s \alpha_1)$ from $\alpha_1 >_s \alpha_2$ and it is not transitive (we cannot derive $\alpha_1 >_s \alpha_3$ from $\alpha_1 >_s \alpha_2$ and $\alpha_2 >_s \alpha_3$). The lack of these properties is the result of the fact that we do not have connected orderings. Moreover, this a-connectedness is crucial for our preference-based deontic logics, see [TvdT96, vdTT97b].

$$
\begin{array}{lll}
\alpha_1 >_s \alpha_2 & =_{def} & \overset{\leftrightarrow}{\Box}(\alpha_1 \rightarrow \Box\neg\alpha_2) \\
O(\alpha|\beta\backslash\gamma) & =_{def} & (\alpha \wedge \beta \wedge \neg\gamma) >_s (\neg\alpha \wedge \beta) \\
 & = & \overset{\leftrightarrow}{\Box}((\alpha \wedge \beta \wedge \neg\gamma) \rightarrow \Box(\beta \rightarrow \alpha)) \\
O^c(\alpha|\beta\backslash\gamma) & =_{def} & (\alpha \wedge \beta \wedge \neg\gamma) >_s (\neg\alpha \wedge \beta) \wedge \overset{\leftrightarrow}{\Diamond}(\alpha \wedge \beta \wedge \neg\gamma) \\
O^{cc}(\alpha|\beta\backslash\gamma) & =_{def} & (\alpha \wedge \beta \wedge \neg\gamma) >_s (\neg\alpha \wedge \beta) \wedge \overset{\leftrightarrow}{\Diamond}(\alpha \wedge \beta \wedge \neg\gamma) \wedge \overset{\leftrightarrow}{\Diamond}(\neg\alpha \wedge \beta)
\end{array}
$$

From the definitions follows immediately the following satisfiability conditions for the modal connectives $\overset{\leftrightarrow}{\Box}$: $M, w \models \overset{\leftrightarrow}{\Box}\alpha$ iff $\forall w' \in W\ M, w' \models \alpha$ and $\overset{\leftrightarrow}{\Diamond}$: $M, w \models \overset{\leftrightarrow}{\Diamond}\alpha$ iff $\exists w' \in W\ M, w' \models \alpha$. As a consequence, the truth value of a contextual obligation does not depend on the world in which the obligation is evaluated. For a model $M = \langle W, \leq, V\rangle$ we have $M \models O(\alpha|\beta\backslash\gamma)$ (i.e. for all worlds $w \in W$ we have $M, w \models O(\alpha|\beta\backslash\gamma)$) iff there is a world $w \in W$ such that $M, w \models O(\alpha|\beta\backslash\gamma)$.

The following proposition shows the truth conditions of contextual obligations.

**Proposition 1 (contextual obligation)** *Let $M = \langle W, \leq, V\rangle$ be a CT4O model and let $|\alpha|$ be the set of worlds that satisfy $\alpha$. For a world $w \in W$, we have $M, w \models O(\alpha|\beta\backslash\gamma)$ iff for all $w_1 \in |\alpha \wedge \beta \wedge \neg\gamma|$ and all $w_2 \in |\neg\alpha \wedge \beta|$ we have $w_2 \not\leq w_1$.*

**Proof** *Follows directly from the definition of $>_s$.*

The following proposition shows several properties of contextual obligations.

**Proposition 2 (theorems of CDL)** *The logic CT4O validates the following theorems.*

| | |
|---|---|
| **SA**: | $O(\alpha|\beta_1\backslash\gamma) \rightarrow O(\alpha|\beta_1 \wedge \beta_2\backslash\gamma)$ |
| **WC**: | $O(\alpha_1 \wedge \alpha_2|\beta\backslash\gamma) \rightarrow O(\alpha_1|\beta\backslash\gamma \vee \neg\alpha_2)$ |
| **WT**: | $O(\alpha|\beta\backslash\gamma_1) \rightarrow O(\alpha|\beta\backslash\gamma_1 \vee \gamma_2)$ |
| **AND**: | $(O(\alpha_1|\beta\backslash\gamma) \wedge O(\alpha_2|\beta\backslash\gamma)) \rightarrow O(\alpha_1 \wedge \alpha_2|\beta\backslash\gamma)$ |
| **RSA**: | $(O^c(\alpha|\beta_1\backslash\gamma) \wedge \overset{\leftrightarrow}{\Diamond}(\alpha \wedge \beta_1 \wedge \beta_2 \wedge \neg\gamma)) \rightarrow O^c(\alpha|\beta_1 \wedge \beta_2\backslash\gamma)$ |
| **RAND**: | $(O^c(\alpha_1|\beta\backslash\gamma) \wedge O^c(\alpha_2|\beta\backslash\gamma) \wedge \overset{\leftrightarrow}{\Diamond}(\alpha_1 \wedge \alpha_2 \wedge \beta \wedge \neg\gamma)) \rightarrow O^c(\alpha_1 \wedge \alpha_2|\beta\backslash\gamma)$ |

**Proof** *The theorems can easily be proven in the preferential semantics. Consider* **WC**. *Assume $M \models O(\alpha_1 \wedge \alpha_2|\beta\backslash\gamma)$. Let $W_1 = |\alpha_1 \wedge \alpha_2 \wedge \beta \wedge \neg\gamma|$ and $W_2 = |\neg(\alpha_1 \wedge \alpha_2) \wedge \beta|$, and $w_2 \not\leq w_1$ for $w_1 \in W_1$ and $w_2 \in W_2$. Moreover, let $W_1' = |\alpha_1 \wedge \beta \wedge \neg(\gamma \vee \neg\alpha_2)|$ and $W_2' = |\neg\alpha_1 \wedge \beta|$. We have $w_2 \not\leq w_1$ for $w_1 \in W_1'$ and $w_2 \in W_2'$, because $W_1 = W_1'$ and $W_2' \subseteq W_2$. Thus, $M \models O(\alpha_1\,|\,\beta\backslash\gamma \vee \neg\alpha_2)$. Verification of the other theorems is left to the reader.*[2]

To illustrate the properties of CDL, we compare it with Bengt Hansson's minimizing dyadic deontic logic. First we recall some well-known definitions and properties of this logic. In Bengt Hansson's classical preference semantics [Han71], as studied by Lewis [Lew74], a dyadic obligation, which we denote by $O_{HL}(\alpha|\beta)$, is true in a model iff 'the minimal (or preferred) $\beta$ worlds satisfy $\alpha$'. A weaker version of this definition, which allows for moral dilemmas, is that $O_{HL}^w(\alpha\,|\,\beta)$ is true in a model iff there is an *equivalence class* of minimal (or preferred) $\beta$ worlds that satisfy $\alpha$.

**Definition 5 (Minimizing obligation)** *Let $M = \langle W, \leq, V\rangle$ be a Kripke model and $|\alpha|$ be the set of all worlds of $W$ that satisfy $\alpha$. The weak Hansson-Lewis obligation '$\alpha$ should be the case if $\beta$ is the case', written as $O_{HL}^w(\alpha|\beta)$, is defined as follows.*

$$O_{HL}^w(\alpha|\beta) \quad =_{def} \quad \overset{\leftrightarrow}{\Diamond}(\beta \wedge \Box(\beta \rightarrow \alpha))$$

[2] This proposition also shows an important advantage of the axiomatisation of the deontic logic in a underlying preference logic: the properties of our dyadic obligations can simply be proven by proving (un)derivability in CT4O.

The model $M$ satisfies the weak Hansson-Lewis obligation '$\alpha$ should be the case if $\beta$ is the case', written as $M \models O^w_{HL}(\alpha \mid \beta)$, iff there is a world $w_1 \in \mid \alpha \wedge \beta \mid$ such that for all $w_2 \in \mid \neg\alpha \wedge \beta \mid$ we have $w_2 \not\leq w_1$. The following proposition shows that the expression $O^w_{HL}(\alpha \mid \beta)$ corresponds to a weak Hansson-Lewis minimizing obligation. For simplicity, we assume that there are no infinite descending chains.

**Proposition 3** *Let $M = \langle W, \leq, V\rangle$ be a CT4O model, such that there are no infinite descending chains. As usual, we write $w_1 < w_2$ for $w_1 \leq w_2$ and not $w_2 \leq w_1$, and $w_1 \sim w_2$ for $w_1 \leq w_2$ and $w_2 \leq w_1$. A world $w$ is a minimal $\beta$-world, written as $M, w \models_{\leq} \beta$, iff $M, w \models \beta$ and for all $w' < w$ holds $M, w' \not\models \beta$. A set of worlds is an equivalence class of minimal $\beta$-worlds, written as $E_\beta$, iff there is a $w$ such that $M, w \models_{\leq} \beta$ and $E_\beta = \{w' \mid M, w' \models \beta$ and $w \sim w'\}$. We have $M \models O^w_{HL}(\alpha|\beta)$ iff there is an $E_\beta$ such that $E_\beta \subseteq|\alpha|$.*

**Proof** $\Leftarrow$ *Follows directly from the definitions. Assume there is a $w$ such that $M, w \models_{\leq} \beta$ and $E_\beta = \{w' \mid M, w' \models \beta$ and $w \sim w'\}$ and $E_\beta \subseteq|\alpha|$. For all $w_2 \in|\neg\alpha \wedge \beta|$ we have $w_2 \not\leq w$.*

$\Rightarrow$ *Assume that there is a world $w_1 \in|\alpha \wedge \beta|$ such that for all $w_2 \in|\neg\alpha \wedge \beta|$ we have $w_2 \not\leq w_1$. Let $w$ be a minimal $\beta$-world such that $M, w \models_{\leq} \beta$ and $w \leq w_1$ (that exists because there are no infinite descending chains), and let $E_\beta = \{w' \mid M, w' \models \beta$ and $w \sim w'\}$.*

Now we are ready to compare our contextual deontic logic with Bengt Hansson's dyadic deontic logic. The following proposition shows that under a certain condition, the contextual obligation $O(\alpha|\beta\backslash\gamma)$ is true in a model if and only if a set of the weak Hansson-Lewis minimizing obligations $O^w_{HL}(\alpha|\beta')$ is true in the model.

**Proposition 4** *Let $M = \langle W, \leq, V\rangle$ be a CT4O model, that has no worlds that satisfy the same propositional sentences. Hence, we identify the set of worlds with a set of propositional interpretations, such that there are no duplicate worlds. We have $M \models O^{cc}(\alpha|\beta\backslash\gamma)$ iff there are $\alpha \wedge \beta \wedge \neg\gamma$ and $\neg\alpha \wedge \beta$ worlds, and for all propositional $\beta'$ such that $M \models \overset{\leftrightarrow}{\Box} (\beta' \rightarrow \beta)$ and $M \not\models \overset{\leftrightarrow}{\Box} (\beta' \rightarrow \gamma)$, we have $M \models O^w_{HL}(\alpha|\beta')$.*

**Proof** $\Rightarrow$ *Follows directly from the semantic definitions.* $\Leftarrow$ *Every world is characterized by a unique propositional sentence. Let $\overline{w}$ denote the sentence that uniquely characterizes world $w$. Proof by contraposition. If $M \not\models O^{cc}(\alpha \mid \beta \backslash \gamma)$, then there are $w_1, w_2$ such that $M, w_1 \models \alpha \wedge \beta \wedge \neg\gamma$ and $M, w_2 \models \neg\alpha \wedge \beta$ and $w_2 \leq w_1$. Choose $\beta' = \overline{w_1} \vee \overline{w_2}$. The world $w_2$ is an element of the preferred $\beta'$ worlds, because there are no duplicate worlds. (If duplicate worlds are allowed, then there could be a $\beta'$ world $w_3$ which is a duplicate of $w_1$, and which is strictly preferred to $w_1$ and $w_2$.) We have $M, w_2 \not\models \alpha$ and therefore $M \not\models O^w_{HL}(\alpha|\beta')$,*

The following example illustrates that contextual deontic logic solves the contrary-to-duty paradoxes.

**Example 3 (Apples-and-Pears, continued)** *Consider the premise set of contextual obligations $S = \{O^c(a \vee p|\top\backslash\bot), O^c(\neg a|\top\backslash\bot)\}$. The crucial observation is that we do not have $O^{cc}(p|a\backslash\gamma)$ for any $\gamma$, and a typical countermodel is the model in Figure 4. This figure should be read as follows. Each circle represents an equivalence class of worlds, that satisfy the propositions written in the circle. The arrows represent strict preferences for all worlds in the circle.*

*We have $S \models O^c(p|\top\backslash a)$, as is shown in Figure 5, which expresses that pears should be bought, unless apples are bought. From the contextual obligation $O^c(p \mid \top \backslash a)$ we cannot derive $O(p \mid a \backslash a)$ due to the unless clause.*

It is easily verified that the contextual obligations also solve the other contrary-to-duty paradoxes in Example 1.

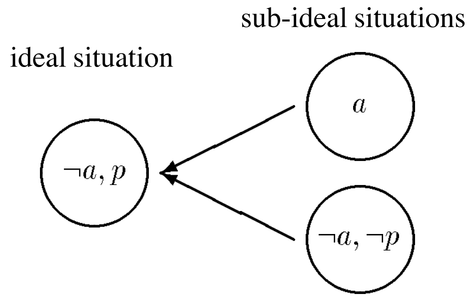


Figure 4: Semantic solution in contextual deontic logic

$$\dfrac{\dfrac{\dfrac{O^c(a \vee p | \top \setminus \bot) \quad O^c(\neg a | \top \setminus \bot)}{O^c(\neg a \wedge p | \top \setminus \bot)}\ \text{AND}}{O^c(p | \top \setminus a)}\ \text{WC}}{O^c(p | a \setminus a)}\ \text{NO (RSA)}$$

Figure 5: Proof-theoretic solution in contextual deontic logic

# 4 Conclusions

Recently, several researchers have noticed a remarkable resemblance between logics of qualitative decision theory, logics of desires and deontic logic, see e.g. [Bou94b, Lan96]. In future research, we will investigate whether contextual deontic logic proposed here can be applied to model qualitative decision theory, and which extensions are needed (see [TvdT96] for possible extensions).

In the introduction, we already observed that we can also define contextual defaults '$\alpha$ is usually the case if $\beta$ is the case unless $\gamma$ is the case,' written as $\beta : \neg\gamma / \alpha$. A distinction between Reiter's default logic and contextual obligations is that the latter has commitment to justifications. Moreover, observe that contextual obligations give rise to a kind of defeasibility, in the sense that the obligations lack unrestricted strengthening of the antecedent (the typical property of defeasible conditionals [Alc93]). However, it is important to notice that this defeasibility related to contextual reasoning is in fundamentally different from the defeasibility related to exceptional circumstances or abnormality, see [vdTT95, vdTT97a].

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