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A General Family of Preferential Belief Removal Operators

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Abstract Most belief change operators in the AGM tradition assume an underlying plausibility ordering over the possible worlds which is transitive and complete. A unifying structure for these operators, based on supplementing the plausibility ordering with a second, guiding, relation over the worlds was presented in Booth et al. (Artif Intell 174:1339–1368, 2010). However it is not always reasonable to assume completeness of the underlying ordering. In this paper we generalise the structure of Booth et al. (Artif Intell 174:1339–1368, 2010) to allow incomparabilities between worlds. We axiomatise the resulting class of belief removal functions, and show that it includes an important family of removal functions based on finite prioritised belief bases.

Keywords Belief revision · Belief removal · Belief contraction · Belief change · Plausibility orderings · Finite belief bases

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1 Introduction

The problem of *belief removal* [1, 7, 23], i.e., the problem of what an agent, hereafter $\mathcal{A}$, should believe after being directed to remove some sentence from his stock of beliefs, has been well studied in philosophy and in AI over the last 25 years. During that time many different families of removal functions have been studied. A great many of them are based on constructions employing *total preorders* over the set of possible worlds which is meant to stand for some notion $\leq$ of relative *plausibility* [14]. A unifying construction for these families was given in [7], in which a general construction was proposed which involved supplementing the relation $\leq$ with a second, guiding, relation $\preceq$ which formed a subset of $\leq$. By varying the conditions on $\preceq$ and its interaction with $\leq$ many of the different families can be captured as instances.

The construction in [7] achieves a high level of generality, but one can argue it fails to be general enough in one important respect: the underlying plausibility order $\leq$ is *always* assumed to be a total preorder which by definition implies it is *complete*, i.e., for any two worlds x, y, we have either $x \leq y$ or $y \leq x$. This implies that agent $\mathcal{A}$ is *always* able to decide which of x, y is more plausible. This is not always realistic, as already argued by Katsuno and Mendelzon [14] who show that a number of intuitively appealing constructions for *belief revision* do not make use of total preorders. Given the strong link between belief removal and belief revision, the same argument applies to belief removal as well. It therefore seems desirable to study belief removal based on plausibility orderings which allow *incomparabilities*. Some work has been done on this ([2, 5, 11, 14, 21], and especially the choice-theoretic approach to belief change advocated in [22]) but not much. This is in contrast to work in nonmonotonic reasoning (NMR), the research area which is so often referred to as the "other side of the coin" to belief change. In NMR, semantic models based on incomplete orderings are the norm, with work dating back to the seminal papers on *preferential models* of [15, 24]. Our aim in this paper is to relax the completeness assumption from [7] and to investigate the resulting, even more general class of removal functions.

The plan of this paper is as follows. In Section 2 we give our generalised definition of the construction from [7], which we call *(semi-modular) contexts*. We describe their associated removal functions, as well as mention the characterisation from [7]. Then in Section 3 we present an axiomatic characterisation of the family of removal functions generated by semi-modular contexts. Then, in Section 4 we mention a couple of further restrictions on contexts, leading to two corresponding extra postulates. In Section 5 we mention an important subfamily of the general family, i.e., those removals which may be generated by a finite prioritised base of *defaults,* before moving on to AGM style removal in Section 6. We conclude in Section 7.

1.1 Preliminaries

We work in a finitely-generated propositional language L. The set of non-tautologous sentences in L is denoted by L_*. The set of propositional worlds/models is W. For any set of sentences $X \subseteq L$, the set of worlds which satisfy every sentence in X is denoted by $[X]$. Classical logical consequence and equivalence are denoted by $\vdash$ and $\equiv$ respectively. As above, we let $\mathcal{A}$ denote some agent whose beliefs are subject to change. A *belief set* for $\mathcal{A}$ is represented by a single sentence which is meant to stand for all its logical consequences. Given that we work in a finitely generated setting, this is a reasonable representation, and one that is frequently adopted.[1] A *belief removal function* (hereafter just *removal function*) belonging to $\mathcal{A}$ is a unary function $\circledast$ which takes any non-tautologous sentence $\lambda \in L_*$ as input and returns a new belief set $\circledast(\lambda)$ for $\mathcal{A}$ such that $\circledast(\lambda) \nvdash \lambda$. For any removal function $\circledast$ we can always derive an associated belief set. It is just the belief set obtained by removing the contradiction, i.e., $\circledast(\bot)$.

The following definitions about orderings will be useful in what follows. A binary relation R over W is:

- *reflexive* iff $\forall x : xRx$
- *transitive* iff $\forall x, y, z : xRy \ \& \ yRz \rightarrow xRz$
- *complete* iff $\forall x, y : xRy \vee yRx$
- a *preorder* iff it is reflexive and transitive
- a *total preorder* iff it is a complete preorder

The above notions are used generally when talking of "weak" orderings, where xRy is meant to stand for something like "x is *at least as good as* y". However in this paper, following the lead of [21], we will find it more natural to work under a *strict* reading, where xRy denotes "x is *strictly better than* y". In this setting, the following notions will naturally arise. R is:

- *irreflexive* iff $\forall x : \text{not}(xRx)$
- *modular* iff $\forall x, y, z : xRy \rightarrow (xRz \vee zRy)$
- a *strict partial order (spo)* iff it is both irreflexive and transitive
- the *strict part of* another relation R' iff $\forall x, y : xRy \leftrightarrow \left(xR'y \ \& \ \text{not}(yR'x)\right)$
- the *converse complement* of R' iff $\forall x, y : xRy \leftrightarrow \text{not}\left(yR'x\right)$

We have that R is a modular spo iff it is the strict part of a total preorder [18]. So in terms of *strict* relations, much of the previous work on belief removal, including [7], assumes an underlying strict order which is a modular spo. It is precisely the modularity condition which we want to relax in this paper.

[1] E.g., this is the representation used by Katsuno and Mendelzon [14].

Given any ordering R and $x \in W$, let $\nabla_R(x) = \{z \in W \mid zRx\}$ be the set of all worlds below x in R. Then we may define a new binary relation $\sqsubseteq^R$ from R by setting $x \sqsubseteq^R y$ iff $\nabla_R(x) \subseteq \nabla_R(y)$. That is, $x \sqsubseteq^R y$ iff every element below x in R is also below y in R. It is easy to check that if R is a modular spo then $x \sqsubseteq^R y$ iff not (yRx), i.e., $\sqsubseteq^R$ is just the converse complement of R.

2 Contexts, Modular Contexts and Removals

In this section we set up our generalised definition of a context, show how each such context yields a removal function and vice versa, and recap the main results from [7].

2.1 Contexts

We assume our agent $\mathcal{A}$ has in his mind *two* binary relations $(<, \prec)$ over the set W. The relation $<$ is a *strict* plausibility relation which forms the basis for $\mathcal{A}$'s actionable beliefs, i.e., $x < y$ means that, to $\mathcal{A}$'s mind, and on the basis of all available evidence, *world x is strictly more plausible than y*. We assume $<$ is a strict partial order. In addition to this there is a second binary relation $\prec$. This relation is open to several different interpretations, but the one we attach is as follows: $x \prec y$ means *"$\mathcal{A}$ has **an explicit reason** to hold x more plausible than y (or to treat x more favourably than y)"*. We will use $\preceq$ to denote the converse complement of $\prec$, i.e., $x \preceq y$ iff $y \not\prec x$. Thus $x \preceq y$ iff $\mathcal{A}$ has no reason to treat y more favourably than x. Note $\preceq$ and $\prec$ are interdefinable, and we find it convenient to switch between them freely.

Note the equivalence "$x \prec y$ iff both $x \preceq y$ and $y \not\preceq x$" holds only if $\prec$ is asymmetric, which might not hold in general, since it is perfectly possible for $\mathcal{A}$ to have one explicit reason to hold x more plausible than y, and another to hold y more plausible than x. In this case both these reasons will compete with each other, with at most one of the pairs $\langle x, y\rangle$ or $\langle y, x\rangle$ making it into $\mathcal{A}$'s plausibility relation $<$.

What are the properties of $\prec$? We assume only two things, at least to begin with: (1) an agent can never possess a reason to hold a world strictly more plausible than itself, and (2) an agent does not hold a world x to be more plausible than another world y, i.e., $x < y$, *without* being in possession of some reason for doing so. (Note this latter property lends a certain "foundationalist" flavour to our construction.) All this is formalised in the following definition:

Definition 2.1 A *context* $\mathcal{C}$ is a pair of binary relations $(<, \prec)$ over W such that:

($\mathcal{C}$**1**) $<$ is a strict partial order
($\mathcal{C}$**2**) $\prec$ is irreflexive
($\mathcal{C}$**3**) $< \subseteq \prec$

If $<$ is modular then we call $\mathcal{C}$ a *modular context.* We will later have grounds for strengthening ($\mathcal{C}$**3**).

Example 2.2 Assume $L = \{p, q\}$ and let the four valuations of L be $W = \{00, 11, 01, 10\}$, where the first and second numbers denote the truth-values of p, q respectively. Then a possible context $\mathcal{C} = (<, \prec)$ could be specified as follows:

$$< = \{(00, 10), (01, 11)\}$$
$$\prec = < \cup \{(00, 11), (01, 00), (10, 00), (10, 11), (11, 10), (11, 00), (11, 01)\}$$

How does $\mathcal{A}$ use his context $\mathcal{C}$ to construct a removal function $\circledast_{\mathcal{C}}$? In terms of models, the set $[\circledast_{\mathcal{C}}(\lambda)]$ of models of his new belief set, when removing a sentence λ, *must* include some $\neg\lambda$-worlds. Following the usual practice in belief revision, he should take the most plausible ones according to $<$, i.e., the $<$-minimal ones. But which, if any, of the λ-worlds should be included? The following principle was proposed by Rott and Pagnucco [23]:

Principle of Weak Preference
If one object is held in equal or higher regard than another, the former should be treated no worse than the latter.

Rott and Pagnucco use this principle to argue that the new set of worlds following removal should contain all worlds x which are not less plausible than a $<$-minimal $\neg\lambda$-world y, i.e., $y \not< x$. We propose to apply a tempered version of this principle using the second ordering $\prec$. We include x if there is *no explicit reason to believe* that y is more plausible than x, i.e., if $y \not\prec x$.

Definition 2.3 ($\circledast$ from $\mathcal{C}$) Given a context $\mathcal{C}$ we define the removal function $\circledast_{\mathcal{C}}$ by setting, for each $\lambda \in L_*$, $[\circledast_{\mathcal{C}}(\lambda)] = \bigcup \left\{\nabla_{\preceq}(y) \mid y \in \min_{<}([\neg\lambda])\right\}$.

Example 2.2 (contd.) Let $\mathcal{C} = (<, \prec)$ be the context given in Example 2.2, and consider the world 00. For $x \in W$ we have $x \in \nabla_{\preceq}(00)$ iff $x \preceq 00$ iff $00 \not\prec x$. Thus $\nabla_{\preceq}(00) = \{00, 01\}$. Similarly we obtain $\nabla_{\preceq}(11) = \{11\}$, $\nabla_{\preceq}(01) = \{10, 01\}$, and $\nabla_{\preceq}(10) = \{10, 01\}$. Now suppose, for example, we want to remove $\neg p \wedge q$ using $\circledast_{\mathcal{C}}$. Then $\min_{<}([\neg(\neg p \wedge q)]) = \{00, 11\}$, so $[\circledast_{\mathcal{C}}(\neg p \wedge q)] = \nabla_{\preceq}(00) \cup \nabla_{\preceq}(11) = \{00, 01, 11\}$. Note that the set of models of the belief set associated to $\circledast_{\mathcal{C}}$, i.e., $\circledast_{\mathcal{C}}(\bot)$, is given by $[\circledast_{\mathcal{C}}(\bot)] = \nabla_{\preceq}(00) \cup \nabla_{\preceq}(01) = \{00, 01, 10\}$.

It can be shown that different contexts give rise to different removal functions, i.e., the mapping $\mathcal{C} \mapsto \circledast_{\mathcal{C}}$ is injective. The case of modular contexts was the one which was studied in detail in [7], where it was shown how, by placing various restrictions on the interaction between $<$ and $\prec$, this family captures a wide range of removal operations which have been previously studied, for example both AGM contraction *and* AGM revision [1],[2] severe withdrawal [23], systematic withdrawal [19] and belief liberation [6]. For the general family in that paper the following representation result was proved.

Theorem 2.4 [7, 8] *Let $\mathcal{C}$ be a modular context. Then $\circledast_{\mathcal{C}}$ satisfies the following rules:*

($\circledast$**1**) $\circledast(\lambda) \nvdash \lambda$
($\circledast$**2**) *If* $\lambda_1 \equiv \lambda_2$ *then* $\circledast(\lambda_1) \equiv \circledast(\lambda_2)$
($\circledast$**3**) *If* $\circledast(\lambda \wedge \chi) \vdash \chi$ *then* $\circledast(\lambda \wedge \chi \wedge \psi) \vdash \chi$
($\circledast$**4**) *If* $\circledast(\lambda \wedge \chi) \vdash \chi$ *then* $\circledast(\lambda \wedge \chi) \vdash \circledast(\lambda)$
($\circledast$**5**) $\circledast(\lambda \wedge \chi) \vdash \circledast(\lambda) \vee \circledast(\chi)$
($\circledast$**6**) *If* $\circledast(\lambda \wedge \chi) \nvdash \lambda$ *then* $\circledast(\lambda) \vdash \circledast(\lambda \wedge \chi)$

Furthermore if $\circledast$ is any removal function satisfying the above six rules, there exists a unique modular context $\mathcal{C}$ such that $\circledast = \circledast_{\mathcal{C}}$.

All these rules are familiar from the belief removal literature. ($\circledast$**1**) is the Success postulate which says the sentence to be removed is no longer implied by the new belief set, while ($\circledast$**2**) is a syntax-irrelevance property. ($\circledast$**3**) is sometimes known as Conjunctive Trisection [13, 21]. A slight reformulation of it can be found already in [1] under the name Partial Antitony. It says if χ is believed after removing the conjunction $\lambda \wedge \chi$, then it should also be believed when removing the longer conjunction $\lambda \wedge \chi \wedge \psi$. Rule ($\circledast$**4**) is closely-related to the rule Cut from the area of non-monotonic reasoning [15], while ($\circledast$**5**) and ($\circledast$**6**) are the two AGM supplementary postulates for contraction [1]. Note we can give ($\circledast$**3**) an equivalent formulation:

Proposition 2.5 *In the presence of* ($\circledast$**2**)*, rule* ($\circledast$**3**) *is equivalent to*

($\circledast$**3′**) $\neg\chi \wedge \circledast(\lambda \wedge \chi) \vdash \circledast(\chi)$

Proof To show ($\circledast$**3**) and ($\circledast$**2**) imply ($\circledast$**3′**), first, we know $\circledast(\chi) \vdash (\neg\chi \rightarrow \circledast(\chi))$. Since $\chi \equiv (\neg\chi \rightarrow \circledast(\chi)) \wedge \chi$ this means $\circledast(\chi) \equiv \circledast((\neg\chi \rightarrow \circledast(\chi)) \wedge \chi)$ by ($\circledast$**2**) and so we obtain $\circledast((\neg\chi \rightarrow \circledast(\chi)) \wedge \chi) \vdash (\neg\chi \rightarrow \circledast(\chi))$. Applying ($\circledast$**3**) to this we may deduce $\circledast((\neg\chi \rightarrow \circledast(\chi)) \wedge \chi \wedge \lambda) \vdash (\neg\chi \rightarrow \circledast(\chi))$. But $(\neg\chi \rightarrow \circledast(\chi)) \wedge \chi \wedge \lambda \equiv \lambda \wedge \chi$. Hence by ($\circledast$**2**) we get $\circledast(\lambda \wedge \chi) \vdash (\neg\chi \rightarrow \circledast(\chi))$, equivalently $\neg\chi \wedge \circledast(\lambda \wedge \chi) \vdash \circledast(\chi)$ as required.

To show ($\circledast$**3′**) and ($\circledast$**2**) imply ($\circledast$**3**), suppose $\circledast(\lambda \wedge \chi) \vdash \chi$. Now, ($\circledast$**3′**) (with a little help from ($\circledast$**2**)) tells us $\neg(\lambda \wedge \chi) \wedge \circledast(\lambda \wedge \chi \wedge \psi) \vdash \circledast(\lambda \wedge \chi)$. Hence using this with the assumption $\circledast(\lambda \wedge \chi) \vdash \chi$ yields $\neg(\lambda \wedge \chi) \wedge \circledast(\lambda \wedge \chi \wedge \psi) \vdash \chi$. By classical logic this is equivalent to the desired $\circledast(\lambda \wedge \chi \wedge \psi) \vdash \chi$. □

[2]The fact that basic removal also covers AGM revision is what motivated our choice of the contraction-revision "hybrid" symbol $\circledast$ to denote removal functions.

As a special case of ($\circledast$**3**$'$) (putting $\lambda = \bot$) we get $\neg\chi \wedge \circledast(\bot) \vdash \circledast(\chi)$, i.e., the set of sentences believed after removing χ is a subset of the set of sentences believed after *expanding* (i.e., conjoining) the initial belief set with $\neg\chi$. (We remark that the preceding proposition was proved, using different notation, in [7]. We include its proof here for completeness.)

Given the prominence of AGM contraction in the literature [1], it is worth noting that this list does not include the AGM contraction postulates Vacuity ($\circledast(\bot) \nvdash \lambda$ implies $\circledast(\lambda) \equiv \circledast(\bot)$), Inclusion ($\circledast(\bot) \vdash \circledast(\lambda)$) and Recovery ($\circledast(\lambda) \wedge \lambda \vdash \circledast(\bot)$), none of which are valid in general for removal functions generated from modular contexts. Vacuity has been argued against as a general principle of belief removal in [7, 8]. Inclusion has been questioned in [6], while Recovery has long been regarded as controversial (see, e.g., [12]). Nevertheless we will see in Section 6 how each of these three rules may be captured within our general framework.

The second part of Theorem 2.4 was proved using the following construction.

Definition 2.6 ($\mathcal{C}$ from $\circledast$) Given any removal function $\circledast$ we define the context $\mathcal{C}(\circledast) = (<, \prec)$ as follows: $x < y$ iff $y \notin [\circledast(\neg x \wedge \neg y)]$ (equivalently, $\circledast(\neg x \wedge \neg y) \vdash \neg y$) and $x \prec y$ iff $y \notin [\circledast(\neg x)]$ (equivalently, $\circledast(\neg x) \vdash \neg y$).[3]

Booth et al. [7] showed that if $\circledast$ satisfies ($\circledast$**1**)–($\circledast$**6**) then $\mathcal{C}(\circledast)$ is a modular context and $\circledast = \circledast_{\mathcal{C}(\circledast)}$.

3 Characterising the General Family

Now we want to drop the assumption that $<$ is modular and assume only it is a strict partial order. How can we characterise the resulting class of removal functions? This is the question we pose in this section, and we provide an answer at the end of the section. To start with, we focus first on establishing which of the postulates from Theorem 2.4 are sound for the general family, modifying our initial construction as and when necessary. Clearly we cannot expect that all the rules remain sound. In particular rule ($\circledast$**6**) is known to depend on the modularity of $<$ and so might be expected to be the first to go. However we might hope to retain weaker versions of it, for instance:

($\circledast$**6a**) If $\circledast(\lambda \wedge \chi) \vdash \chi$ then $\circledast(\lambda) \vdash \circledast(\lambda \wedge \chi)$
($\circledast$**6b**) $\circledast(\lambda) \wedge \circledast(\chi) \vdash \circledast(\lambda \wedge \chi)$

These two rules appear respectively as ($-$8c) and ($-$8r) in [22] (see also [2]). ($\circledast$**6b**) follows from ($\circledast$**6**) given ($\circledast$**1**).

[3] When a world appears in the scope of a propositional connective, it should be understood as denoting any sentence which has that world as its only model.

Proposition 3.1 *If $\mathcal{C}$ is a general context then $\circledast_{\mathcal{C}}$ satisfies* ($\circledast$**1**), ($\circledast$**2**), ($\circledast$**4**), ($\circledast$**5**) *and* ($\circledast$**6a**).

Proof ($\circledast$**1**) holds because $[\circledast_{\mathcal{C}}(\lambda)]$ picks up at least one $\neg\lambda$-world. ($\circledast$**2**) holds because the result depends only on $[\neg\lambda]$.

To show ($\circledast$**4**) suppose $\circledast_{\mathcal{C}}(\lambda \wedge \chi) \vdash \chi$. Suppose $x \in [\circledast_{\mathcal{C}}(\lambda \wedge \chi)]$. We must show $x \in [\circledast_{\mathcal{C}}(\lambda)]$. But if $x \in [\circledast_{\mathcal{C}}(\lambda \wedge \chi)]$ then $x \preceq y$ for some $y \in \min_<([\neg(\lambda \wedge \chi)])$. It suffices to show $y \in \min_<([\neg\lambda])$. First we need to show $y \in [\neg\lambda]$. Since $\min_<([\neg(\lambda \wedge \chi)]) \subseteq [\circledast_{\mathcal{C}}(\lambda \wedge \chi)]$ by reflexivity of $\preceq$ we know $y \in [\circledast_{\mathcal{C}}(\lambda \wedge \chi)]$ and so, from the assumption $\circledast_{\mathcal{C}}(\lambda \wedge \chi) \vdash \chi$, $y \in [\chi]$. Now since $y \in \min_<([\neg(\lambda \wedge \chi)])$ we also know $y \in [\neg(\lambda \wedge \chi)]$ and so from this and $y \in [\chi]$ we may deduce $y \in [\neg\lambda]$ as required. Now if it were the case that $y \notin \min_<([\neg\lambda])$ then we would have $z < y$ for some $z \in [\neg\lambda]$. But since in this case also $z \in [\neg(\lambda \wedge \chi)]$, this would contradict $y \in \min_<([\neg(\lambda \wedge \chi)])$. Hence $y \in \min_<([\neg\lambda])$ as required.

($\circledast$**5**) holds because of the property $\min_<([\neg\lambda \vee \neg\chi]) \subseteq (\min_<([\neg\lambda]) \cup \min_<([\neg\chi]))$ (which, incidentally, as with the proof of ($\circledast$**4**), doesn't depend on any property of $<$).

For ($\circledast$**6a**) suppose $\circledast_{\mathcal{C}}(\lambda \wedge \chi) \vdash \chi$ and let $x \in [\circledast_{\mathcal{C}}(\lambda)]$. We must show $x \in [\circledast_{\mathcal{C}}(\lambda \wedge \chi)]$. Since $x \in [\circledast_{\mathcal{C}}(\lambda)]$ there exists $y \in \min_<([\neg\lambda])$ such that $x \preceq y$. If we can show $y \in \min_<([\neg(\lambda \wedge \chi)])$ then we are done. So suppose for contradiction $y \notin \min_<([\neg(\lambda \wedge \chi)])$. Then there exists $z \in [\neg(\lambda \wedge \chi)]$ such that $z < y$. By transitivity of $<$ we may assume $z \in \min_<([\neg(\lambda \wedge \chi)])$ Then using the assumption $\circledast_{\mathcal{C}}(\lambda \wedge \chi) \vdash \chi$ we deduce $z \in [\chi]$, which together with the fact $z \in [\neg(\lambda \wedge \chi)]$ implies $z \in [\neg\lambda]$. But $z \in [\neg\lambda]$ and $z < y$ contradicts $y \in \min_<([\neg\lambda])$. Hence $y \in \min_<([\neg(\lambda \wedge \chi)])$ as required. □

($\circledast$**6b**) (and hence ($\circledast$**6**)) does not hold (if L contains more than two propositional variables), as the following counterexample shows:

Example 3.2 Suppose $L = \{p, q, r\}$. We assume each of the eight valuations of L may be represented as a sequence abc where each of a, b and c is an element of $\{0, 1\}$ denoting the truth-value of p, q, r respectively. Let $<= \{(101, 010), (011, 100)\}$ and let $\preceq = \{(111, 010), (111, 100)\}$ (strictly speaking the reflexive closure of this). Recalling that $\preceq$ is the converse complement of $\prec$, one may verify that $(<, \prec)$ so defined forms a context according to Definition 2.1. Then $010 \in \min_<([\neg p])$ so $111 \in [\circledast_{\mathcal{C}}(p)]$. Also $100 \in \min_<([\neg q])$ so $111 \in [\circledast_{\mathcal{C}}(q)]$. Hence $111 \in [\circledast_{\mathcal{C}}(p) \wedge \circledast_{\mathcal{C}}(q)]$. If ($\circledast$**6b**) holds then we would conclude $111 \in [\circledast_{\mathcal{C}}(p \wedge q))]$. But since 010 and 100 are the only two elements x such that $111 \preceq x$ this would imply either $010 \in \min_<([\neg(p \wedge q)])$ or $100 \in \min_<([\neg(p \wedge q)])$. But 010 and 100 are $<$-dominated in $[\neg(p \wedge q)]$ by 101 and 011 respectively. Hence $111 \notin [\circledast_{\mathcal{C}}(p \wedge q))]$ and so ($\circledast$**6b**) cannot hold.

Given the more general scenario in which we now find ourselves, we also lose ($\circledast$**3**), equivalently ($\circledast$**3**$'$), as the following counterexample shows:

Example 3.3 Assume $L = \{p, q\}$ and let the four valuations of L be $W = \{00, 11, 01, 10\}$, where the first and second numbers denote the truth-values of p, q respectively. Let $< = \{(00, 10)\}$ and let $\preceq = \{(10, 01)\}$ (strictly speaking the reflexive closure of this). Again one can verify that $(<, \prec)$ forms a legal context. We have $[\circledast_{\mathcal{C}}(p \wedge q)] = \{00, 10, 01\}$ and $[\circledast_{\mathcal{C}}(q)] = \{00\}$. Hence $10 \in [\neg q \wedge \circledast_{\mathcal{C}}(p \wedge q)]$ but $10 \notin [\circledast_{\mathcal{C}}(q)]$.

This leaves us with a problem, since whereas ($\circledast$**6**) is to be considered somewhat dispensible, ($\circledast$**3**) is a very reasonable property for removal functions. Is there some way we can capture it? It turns out we can capture it if we strengthen the basic property ($\mathcal{C}$**3**) to:

($\mathcal{C}$**3a**) $\quad \preceq \subseteq \sqsubseteq^{<}$[4]

In other words if $z < x$ and $x \preceq y$ then $z < y$. ($\mathcal{C}$**3a**) is a *coherence* condition between $\prec$ and $<$. It is saying that if there is a world z which $\mathcal{A}$ judges to be more plausible than x but not to y then $\mathcal{A}$ has a reason to treat y more favourably than x. Note that for modular contexts ($\mathcal{C}$**3**) and ($\mathcal{C}$**3a**) are equivalent, but in general they are not.

Proposition 3.4 *If $\mathcal{C}$ satisfies* ($\mathcal{C}$**3a**) *then $\circledast_{\mathcal{C}}$ satisfies* ($\circledast$**3**).

Proof To see this suppose $\circledast_{\mathcal{C}}(\lambda \wedge \chi) \vdash \chi$ and let $x \in [\circledast_{\mathcal{C}}(\lambda \wedge \chi \wedge \psi)]$. We must show $x \in [\chi]$. If $x \in [\lambda \wedge \chi]$ then clearly we are done, so suppose $x \in [\neg(\lambda \wedge \chi)]$. We will show $x \in \min_<([\neg(\lambda \wedge \chi)])$, from which we can then deduce $x \in [\chi]$ from $\circledast_{\mathcal{C}}(\lambda \wedge \chi) \vdash \chi$. Suppose for contradiction $x \notin \min_<([\neg(\lambda \wedge \chi)])$. Then there exists $z \in [\neg(\lambda \wedge \chi)]$ such that $z < x$. Now since $x \in [\circledast_{\mathcal{C}}(\lambda \wedge \chi \wedge \psi)]$ there exists $y \in \min_<([\neg(\lambda \wedge \chi \wedge \psi)])$ such that $x \preceq y$. From ($\mathcal{C}$**3a**) this gives $z < y$, but this contradicts the minimality of y, giving the required contradiction. $\square$

Thus ($\mathcal{C}$**3a**) seems necessary. And in fact without it we don't get the following important technical result, which provides the means to describe $<$-minimal λ-worlds purely in terms of the removal function:

Proposition 3.5 *Let $\mathcal{C} = (<, \prec)$ be any context which satisfies* ($\mathcal{C}$**3a**). *Then for all λ such that $\neg\lambda \in L_*$ we have $[\circledast_{\mathcal{C}}(\neg\lambda) \wedge \lambda] = \min_<([\lambda])$.*

[4]Recall from Section 1.1 that $\sqsubseteq^{<}$ is defined as: $x \sqsubseteq^{<} y$ iff $\nabla_<(x) \subseteq \nabla_<(y)$.

Proof The right–to–left inclusion follows from the reflexivity of $\preceq$. For the left–to–right inclusion suppose $x \in [\circledast_{\mathcal{C}}(\neg\lambda) \wedge \lambda]$. Then $x \in [\lambda]$ and $x \preceq y$ for some $y \in \min_{<}([\lambda])$. If there were some $z \in [\lambda]$ such that $z < x$ then we would have $z < y$ from ($\mathcal{C}$**3a**)—contradiction. Hence $x \in \min_{<}([\lambda])$ as required. □

Example 3.3 provides a counterexample showing this might not be possible in general, for there we have $\left[\circledast_{\mathcal{C}}(p \wedge q) \wedge \neg(p \wedge q)\right] = \{00, 10, 01\}$ but $\min_{<}\left(\left[\neg(p \wedge q)\right]\right) = \{00, 01\}$. Note that rule ($\mathcal{C}$**3a**) may also be interpreted as a restricted form of modularity for $<$, since it may be re-written as $\forall x, y, z\, (z < x \rightarrow (y \prec x \vee z < y))$. For this reason we consider the following definition:

Definition 3.6 A *semi-modular context* is any context $\mathcal{C}$ satisfying ($\mathcal{C}$**3a**).

In the rest of the paper we will work only with semi-modular contexts. Example 3.2 shows that $\circledast_{\mathcal{C}}$ still fails in general to satisfy ($\circledast$**6b**) even for semi-modular contexts.

So far we have a list of sound properties for the removal functions defined from semi-modular contexts. They are the same as the rules which characterise modular removal, but with ($\circledast$**6**) replaced by the weaker ($\circledast$**6a**). It might be hoped that this list is complete, i.e., that *any* removal function $\circledast$ satisfying these six rules is equal to $\circledast_{\mathcal{C}}$ for some semi-modular context $\mathcal{C}$. Indeed we might expect to be able to show $\circledast = \circledast_{\mathcal{C}(\circledast)}$, where $\mathcal{C}(\circledast)$ is the context defined via Definition 2.6. The following result gives us a good start.

Proposition 3.7 *Let* $\circledast$ *be any removal function satisfying* ($\circledast$**1**)-($\circledast$**5**) *and* ($\circledast$**6a**). *Then* $\mathcal{C}(\circledast)$ *is a context, i.e., satisfies* ($\mathcal{C}$**1**)–($\mathcal{C}$**3**).

Proof (Outline) $<$ is a strict partial order from ($\circledast$**1**), ($\circledast$**2**), ($\circledast$**3**) and ($\circledast$**6a**). $\prec$ is irreflexive from ($\circledast$**1**), and ($\mathcal{C}$**3**) is satisfied via the following consequence of ($\circledast$**6a**):

($\circledast$**6c**) If $\circledast(\lambda \wedge \chi) \vdash \chi$ then $\circledast(\lambda) \vdash \chi$

Note the antecedent of this rule corresponds to the usual reading of "χ is *strictly more entrenched* than λ". Thus a possible reading of ($\circledast$**6c**) is "χ can never be excluded by removing a strictly less entrenched sentence". □

However to get ($\mathcal{C}$**3a**) it seems an extra property is needed:

($\circledast$**C**) If $\circledast(\lambda) \wedge \neg\lambda \vdash \circledast(\chi) \wedge \neg\chi$ then $\circledast(\lambda) \vdash \circledast(\chi)$

We can rephrase this using the *Levi Identity* [17]. Given any removal function $\circledast$ we may define a *revision function* $\circledast^{\mathrm{R}}$ by setting, for each consistent sentence $\lambda \in L$, $\circledast^{\mathrm{R}}(\lambda) = \circledast(\neg\lambda) \wedge \lambda$. Then rule ($\circledast$**C**) may be equivalently written as:

($\circledast$**C′**) If $\circledast^{\mathrm{R}}(\neg\lambda) \vdash \circledast^{\mathrm{R}}(\neg\chi)$ then $\circledast(\lambda) \vdash \circledast(\chi)$

Thus ($\circledast\mathbf{C}'$) is effectively saying that if revising by $\neg\lambda$ leads to a stronger belief set than revising by $\neg\chi$, then removing λ leads to a stronger belief set than removing χ. The next result confirms that this rule is sound for the removal functions generated by semi-modular contexts, and that this property is enough to show that $\mathcal{C}(\circledast)$ satisfies ($\mathcal{C}\mathbf{3a}$).

Proposition 3.8 *Let $\mathcal{C}$ be a semi-modular context. Then $\circledast_{\mathcal{C}}$ satisfies* ($\circledast\mathbf{C}$). *Furthermore if $\circledast$ is any removal function satisfying* ($\circledast\mathbf{C}$), ($\circledast\mathbf{1}$), ($\circledast\mathbf{3}$) *and* ($\circledast\mathbf{6a}$) *then the context $\mathcal{C}(\circledast)$ satisfies* ($\mathcal{C}\mathbf{3a}$).

Proof To show $\circledast_{\mathcal{C}}$ satisfies ($\circledast\mathbf{C}$), suppose $\circledast_{\mathcal{C}}(\lambda) \wedge \neg\lambda \vdash \circledast_{\mathcal{C}}(\chi) \wedge \neg\chi$. Then by Proposition 3.5 $\min_<([\neg\lambda]) \subseteq \min_<([\neg\chi])$. So if $x \preceq y$ for some $y \in \min_<([\neg\lambda])$ then $x \preceq y$ for some $y \in \min_<([\neg\chi])$. Hence $[\circledast_{\mathcal{C}}(\lambda)] \subseteq [\circledast_{\mathcal{C}}(\chi)]$ as required.

Now let $\circledast$ be a removal function satisfying ($\circledast\mathbf{C}$). We will show that if $z < x$ and $z \not< y$ then $y \prec x$. So suppose $z < x$ and $z \not< y$, i.e., $x \notin [\circledast(\neg z \wedge \neg x)]$ and $y \in [\circledast(\neg z \wedge \neg y)]$. We must show $x \notin [\circledast(\neg y)]$. We know by ($\circledast\mathbf{1}$) that $y \in [\circledast(\neg y)]$. Using this together with $y \in [\circledast(\neg z \wedge \neg y)]$ allows us to deduce $\circledast(\neg y) \wedge y \vdash \circledast(\neg z \wedge \neg y) \wedge \neg(\neg z \wedge \neg y)$ and so, using ($\circledast\mathbf{C}$), we get $\circledast(\neg y) \vdash \circledast(\neg z \wedge \neg y)$. Hence if we can show $\circledast(\neg z \wedge \neg y) \vdash \neg x$ then our desired conclusion $x \notin [\circledast(\neg y)]$ will follow. But from the assumption $x \notin [\circledast(\neg z \wedge \neg x)]$ we know $\circledast(\neg z \wedge \neg x) \vdash \neg x$ and so from this and ($\circledast\mathbf{3}$) we have $\circledast(\neg z \wedge \neg x \wedge \neg y) \vdash \neg x$. From this and ($\circledast\mathbf{6a}$) we get $\circledast(\neg z \wedge \neg y) \vdash \circledast(\neg z \wedge \neg x \wedge \neg y)$ and so the transitivity of $\vdash$ gives the required $\circledast(\neg z \wedge \neg y) \vdash \neg x$. □

Rule ($\circledast\mathbf{C}$) is actually quite strong. In the presence of ($\circledast\mathbf{3}$) it can be shown to imply ($\circledast\mathbf{4}$). This means that, in the axiomatisation of $\circledast_{\mathcal{C}}$ we can replace ($\circledast\mathbf{4}$) with ($\circledast\mathbf{C}$). To show that the list of rules is complete, it remains to prove $\circledast = \circledast_{\mathcal{C}(\circledast)}$. It turns out that here we need the following weakening of ($\circledast\mathbf{6b}$):

($\circledast\mathbf{E}$) $\quad \neg(\lambda \wedge \chi) \wedge \circledast(\lambda) \wedge \circledast(\chi) \vdash \circledast(\lambda \wedge \chi)$

This rule may be reformulated as "$\circledast(\lambda) \wedge \circledast(\chi) \vdash (\lambda \wedge \chi) \vee \circledast(\lambda \wedge \chi)$". In this reformulation, the right hand side of the turnstile may be thought of as standing for all those consequences of the conjunction $\lambda \wedge \chi$ which are *believed* upon its removal. The rule is saying that any such surviving consequence must be derivable from the *combination* of $\circledast(\lambda)$ and $\circledast(\chi)$.

Proposition 3.9 *Let $\mathcal{C}$ be a semi-modular context. Then $\circledast_{\mathcal{C}}$ satisfies* ($\circledast\mathbf{E}$).

Proof Suppose $x \in [\neg(\lambda \wedge \chi) \wedge \circledast_{\mathcal{C}}(\lambda) \wedge \circledast_{\mathcal{C}}(\chi)]$. We must show $x \in [\circledast_{\mathcal{C}}(\lambda \wedge \chi)]$. Since $x \in [\circledast_{\mathcal{C}}(\lambda)]$ we know $x \preceq y_1$ for some $y_1 \in \min_<([\neg\lambda])$. Similarly since $x \in [\circledast_{\mathcal{C}}(\chi)]$ we know $x \preceq y_2$ for some $y_2 \in \min_<([\neg\chi])$. We will show the existence of y_1, y_2 implies $x \in \min_<([\neg(\lambda \wedge \chi)])$, which will be enough to show $x \in [\circledast_{\mathcal{C}}(\lambda \wedge \chi)]$ by Proposition 3.5. Suppose for contradiction $x \notin \min_<([\neg(\lambda \wedge \chi)])$. Then there exists $z < x$ such that $z \in [\neg(\lambda \wedge \chi)]$. Clearly either $z \in [\neg\lambda]$ or $z \in [\neg\chi]$. In the former case we get $z < x \preceq y_1$ and so $z < y_1$ by ($\mathcal{C}\mathbf{3}$)—contradicting the minimality of y_1. Similarly in the latter case we obtain $z < y_2$, contradicting the minimality of y_2. Hence $x \in \min_<([\neg(\lambda \wedge \chi)])$ as required. □

Lemma 3.10 *Any removal function* $\circledast$ *satisfying* ($\circledast$**2**) *and* ($\circledast$**E**) *also satisfies the following rule.*
($\circledast$**E**$'$) *If* $\neg\psi \vdash \circledast(\lambda \wedge \psi) \wedge \circledast(\chi \wedge \psi)$ *then* $\neg\psi \vdash \circledast(\lambda \wedge \chi \wedge \psi)$.

Proof By ($\circledast$**E**) we know $\neg(\lambda \wedge \chi \wedge \psi) \wedge \circledast(\lambda \wedge \psi) \wedge \circledast(\chi \wedge \psi) \vdash \circledast(\lambda \wedge \chi \wedge \psi)$. Since $\neg\psi \vdash \neg(\lambda \wedge \chi \wedge \psi)$ and $\neg\psi \vdash \circledast(\lambda \wedge \psi) \wedge \circledast(\chi \wedge \psi)$ by assumption, this gives the required conclusion. □

Theorem 3.11 *Let* $\circledast$ *be any removal function satisfying* ($\circledast$**1**), ($\circledast$**2**), ($\circledast$**3**), ($\circledast$**C**), ($\circledast$**5**), ($\circledast$**6a**) *and* ($\circledast$**E**). *Then* $\circledast_{\mathcal{C}(\circledast)} = \circledast$.

Proof Let $\lambda \in L_*$. We need to show $[\circledast(\lambda)] = \{x \mid x \preceq y \text{ for some } y \in \min_<([\neg\lambda])\}$.

For the left–to–right inclusion, suppose $x \in [\circledast(\lambda)]$. Let $\min_<([\neg\lambda]) = \{y_1, \ldots, y_k\}$ and $[\neg\lambda] \setminus \min_<([\neg\lambda]) = \{z_1, \ldots, z_l\}$. We will show that $\circledast(\lambda) \vdash \circledast(\bigwedge_{i=1}^k \neg y_i)$. If we can show this then $\circledast(\lambda) \vdash \bigvee_{i=1}^k \circledast(\neg y_i)$ will follow from ($\circledast$**5**), and then from $x \in [\circledast(\lambda)]$ we can deduce $x \in [\circledast(\neg y_i)]$ for some i, i.e., $x \preceq y$ for some $y \in \min_<([\neg\lambda])$ as required. So, to show $\circledast(\lambda) \vdash \circledast(\bigwedge_{i=1}^k \neg y_i)$ first note that $\lambda \equiv (\bigwedge_{i=1}^k \neg y_i) \wedge (\bigwedge_{i=1}^l \neg z_i)$, hence what we need to show is:

$$\circledast\left(\left(\bigwedge_{i=1}^k \neg y_i\right) \wedge \left(\bigwedge_{i=1}^l \neg z_i\right)\right) \vdash \circledast\left(\bigwedge_{i=1}^k \neg y_i\right).$$

This will follow by ($\circledast$**4**), provided we can show

$$\circledast\left(\left(\bigwedge_{i=1}^k \neg y_i\right) \wedge \left(\bigwedge_{i=1}^l \neg z_i\right)\right) \vdash \bigwedge_{i=1}^l \neg z_i. \tag{1}$$

Let $m \in \{1, \ldots, l\}$. Since $z_m \notin \min_<([\neg\lambda])$ we know there must be some $j \in \{1, \ldots, k\}$ such that $y_j < z_m$, i.e., $\circledast(\neg y_j \wedge \neg z_m) \vdash \neg z_m$. But then by ($\circledast$**3**)

$$\circledast\left(\left(\bigwedge_{i=1}^k \neg y_i\right) \wedge \left(\bigwedge_{i=1}^l \neg z_i\right)\right) \vdash \neg z_m.$$

Since this holds for each $m = 1, \ldots, l$ we obtain Eq. 1 as required.

To show the right–to–left inclusion, suppose $x \preceq y$ for some $y \in \min_<([\neg\lambda])$. Then $\circledast(\neg y) \nvdash \neg x$ and $\circledast(\neg y \wedge \neg z) \nvdash \neg y$ for all $z \in [\neg\lambda]$. We want to show $\circledast(\lambda) \nvdash \neg x$. This will be proved if we can show $\circledast(\neg y) \vdash \circledast(\lambda)$. By ($\circledast$**C**) this will be proved if we can show $y \wedge \circledast(\neg y) \vdash \neg\lambda \wedge \circledast(\lambda)$ which, since $y \vdash \circledast(\neg y)$ (by ($\circledast$**1**)) and $y \in [\neg\lambda]$, is equivalent to showing $y \vdash \circledast(\lambda)$. We know $y \vdash \circledast(\neg y \wedge \neg z)$ for all $z \in [\neg\lambda]$. By repeated use of ($\circledast$**E**$'$) we obtain $y \vdash \circledast(\bigwedge_{z \in [\neg\lambda]} \neg z)$. By ($\circledast$**2**) this is equivalent to the required $y \vdash \circledast(\lambda)$. □

Thus, to summarise, the family of removal functions defined from semi-modular contexts is completely characterised by the following rules:

($\circledast$**1**)	$\circledast(\lambda) \nvdash \lambda$
($\circledast$**2**)	If $\lambda_1 \equiv \lambda_2$ then $\circledast(\lambda_1) \equiv \circledast(\lambda_2)$
($\circledast$**3**)	If $\circledast(\lambda \wedge \chi) \vdash \chi$ then $\circledast(\lambda \wedge \chi \wedge \psi) \vdash \chi$
($\circledast$**C**)	If $\circledast(\lambda) \wedge \neg\lambda \vdash \circledast(\chi) \wedge \neg\chi$ then $\circledast(\lambda) \vdash \circledast(\chi)$
($\circledast$**5**)	$\circledast(\lambda \wedge \chi) \vdash \circledast(\lambda) \vee \circledast(\chi)$
($\circledast$**6a**)	If $\circledast(\lambda \wedge \chi) \vdash \chi$ then $\circledast(\lambda) \vdash \circledast(\lambda \wedge \chi)$
($\circledast$**E**)	$\neg(\lambda \wedge \chi) \wedge \circledast(\lambda) \wedge \circledast(\chi) \vdash \circledast(\lambda \wedge \chi)$

In the next section we will later look at a few more reasonable postulates which are not covered by the above list.

4 Transitivity and Priority

In this section we look at imposing an extra couple of properties on semi-modular contexts $\mathcal{C} = (<, \prec)$, both of which were investigated in the case of modular contexts in [7]. There it was shown how the resulting classes of removal functions still remain general enough to include a great many of the classes which have been previously proposed in the context of modular removal.

The first property is the transitivity of $\preceq$, thus making $\preceq$ a preorder. (Recall $\preceq$ is the converse complement of $\prec$, so this is equivalent to making $\prec$ modular.) According to our above interpretation of $\preceq$ this means *if there is no reason to treat y more favourably than x, and no reason to treat z more favourably than y then there is no reason to treat z more favourably than x.*

Proposition 4.1

(1) *If $\preceq$ is transitive then $\circledast_{\mathcal{C}}$ satisfies the following strengthening of* ($\circledast$**C**)*:*

($\circledast$**C+**) *If* $\circledast(\lambda) \wedge \neg\lambda \vdash \circledast(\chi)$ *then* $\circledast(\lambda) \vdash \circledast(\chi)$

(2) *If $\circledast$ satisfies* ($\circledast$**C+**) *and* ($\circledast$**1**) *then the relation $\preceq$ in $\mathcal{C}(\circledast)$ is transitive.*

Proof (1) Suppose $\circledast_{\mathcal{C}}(\lambda) \wedge \neg\lambda \vdash \circledast_{\mathcal{C}}(\chi)$ and let $x \in [\circledast_{\mathcal{C}}(\lambda)]$. We must show $x \in [\circledast_{\mathcal{C}}(\chi)]$. But from $x \in [\circledast_{\mathcal{C}}(\lambda)]$ we know $x \preceq y$ for some $y \in \min_<([\neg\lambda])$. By Proposition 3.5, $y \in [\circledast_{\mathcal{C}}(\lambda) \wedge \neg\lambda]$. From the assumption $\circledast_{\mathcal{C}}(\lambda) \wedge \neg\lambda \vdash \circledast_{\mathcal{C}}(\chi)$ we deduce $y \in [\circledast_{\mathcal{C}}(\chi)]$, i.e., $y \preceq z$ for some $z \in \min_<([\neg\chi])$. Then by transitivity of $\preceq$ we get also $x \preceq z$ and so $x \in [\circledast_{\mathcal{C}}(\chi)]$ as required.

(2) Suppose $x \in [\circledast(\neg y)]$ and $y \in [\circledast(\neg z)]$. We must show $x \in [\circledast(\neg z)]$. But from $y \in [\circledast(\neg z)]$ and $y \vdash \circledast(\neg y)$ (($\circledast$**1**)) we know $\circledast(\neg y) \wedge y \vdash \circledast(\neg z)$. Applying ($\circledast$**C+**) to this yields $\circledast(\neg y) \vdash \circledast(\neg z)$ from which we obtain the required $x \in [\circledast(\neg z)]$ from $x \in [\circledast(\neg y)]$. □

This property is a notational variant of the property (called **(BTran)** there) which was used to characterise transitivity of $\preceq$ in the modular context in [7]. It can be re-written as: If $\circledast^{R}(\neg\lambda) \vdash \circledast(\chi)$ then $\circledast(\lambda) \vdash \circledast(\chi)$. It says that if the belief set following removal of χ is contained in the belief set following the *revision* by $\neg\lambda$, then it must be contained also in the belief set following the removal of λ. This seems like a reasonable property.

Corollary 4.2 *For any removal function $\circledast$, the following are equivalent:*

(1) *$\circledast$ is generated by a semi-modular context $\mathcal{C} = (<, \prec)$ such that $\preceq$ is transitive.*
(2) *$\circledast$ satisfies the list of rules given at the end of Section 3, with ($\circledast$**C**) replaced by ($\circledast$**C**+).*

Now consider the following property of a context $\mathcal{C} = (<, \prec)$:

($\mathcal{C}$**P**) If $x \prec y$ and $y \not\prec x$ then $x < y$

This, too, looks reasonable: if $\mathcal{A}$ has an explicit reason to hold x more plausible than y, but not vice versa, then in the final reckoning he should hold x to be strictly more plausible than y. Consider the following property of removal functions:

($\circledast$**P**) If $\circledast(\lambda) \vdash \chi$ and $\circledast(\chi) \nvdash \lambda$ then $\circledast(\lambda \wedge \chi) \vdash \chi$

This property is briefly mentioned as *Priority* in [5], and is also briefly mentioned right at the end of [10]. It can be read as saying that if λ is excluded following removal of χ, but not vice versa, then χ is strictly more entrenched than λ. **For the case of modular removal**, we can obtain the following exact correspondence between ($\mathcal{C}$**P**) and ($\circledast$**P**) (note this result was proved, in a different notational setting, in [7], where ($\circledast$**P**) is known as **(BPriority)**. We again include the proof for completeness.):

Proposition 4.3

(1) *If $\mathcal{C}$ is a modular context satisfying ($\mathcal{C}$**P**) then $\circledast_{\mathcal{C}}$ satisfies ($\circledast$**P**).*
(2) *If $\circledast$ satisfies ($\circledast$**P**) then $\mathcal{C}(\circledast)$ satisfies ($\mathcal{C}$**P**).*

Proof

(1) Suppose $\circledast_{\mathcal{C}}(\lambda) \vdash \chi$, $\circledast_{\mathcal{C}}(\chi) \nvdash \lambda$ and, for contradiction, $\circledast_{\mathcal{C}}(\lambda \wedge \chi) \nvdash \chi$. Then from $\circledast_{\mathcal{C}}(\chi) \nvdash \lambda$ we know there exist x, y such that $x \in [\neg\lambda]$, $x \preceq y$ and $y \in \min_<([\neg\chi])$, while from $\circledast_{\mathcal{C}}(\lambda \wedge \chi) \nvdash \chi$ we know there exist z, w such that $z \in [\neg\chi]$, $z \preceq w$ and $w \in \min_<([\neg(\lambda \wedge \chi)])$. We have the following chain of inequalities:

$$x \not< w \not< z \not< y \not< x. \qquad (2)$$

To see this, note that $x \not< w$ follows from $x \in [\neg\lambda]$ and the minimality of w while $z \not< y$ follows from $z \in [\neg\chi]$ and the minimality of y. The other two inequalities $w \not< z$ and $y \not< x$ come respectively from $z \preceq w$ and $x \preceq y$ and the fact $< \subseteq \prec$. Since $<$ is modular, which equivalently means $\not<$ is transitive, this chain of inequalities yields $x \not< y$. Using this with $x \preceq y$ and ($\mathcal{C}$**P**) yields $y \preceq x$. Now, Eq. 2 above also gives $w \not< x$, which using the minimality of w and the modularity of $<$ is enough to show $x \in \min_<([\neg\lambda])$. Hence $y \in [\circledast_\mathcal{C}(\lambda)]$. But $y \in [\neg\chi]$, contradicting $\circledast_\mathcal{C}(\lambda) \vdash \chi$.

(2) We will show $x \preceq y$ and $y \not\preceq x$ implies $x < y$. So suppose $x \preceq y$ and $y \not\preceq x$, i.e., by definition of $\mathcal{C}(\circledast)$, $\circledast(\neg y) \nvdash \neg x$ and $\circledast(\neg x) \vdash \neg y$. Then applying ($\circledast$**P**) to this gives $\circledast(\neg x \wedge \neg y) \vdash \neg y$, i.e., $x < y$ as required. □

The proof of Proposition 4.3(1) makes critical use of the modularity of $<$. It turns out that ($\circledast$**P**) is *not* sound for general semi-modular contexts, even if we insist on ($\mathcal{C}$**P**).

Example 4.4 Suppose $L = \{p, q\}$ and that $<= \{(01, 11)\}$ while $\preceq= \{(01, 11)\}$ (strictly speaking the reflexive closure of this). One can verify that $\mathcal{C}$ is a semi-modular context and that ($\mathcal{C}$**P**) is satisfied. Now let $\lambda = p \vee \neg q$ and $\chi = \neg p$. Then $[\circledast_\mathcal{C}(\lambda)] = \{01\}$, $[\circledast_\mathcal{C}(\chi)] = \{11, 01, 10\}$ and $[\circledast_\mathcal{C}(\lambda \wedge \chi)] = \{01, 10\}$ and we have $\circledast_\mathcal{C}(\lambda) \vdash \chi$, $\circledast_\mathcal{C}(\chi) \nvdash \lambda$, and $\circledast_\mathcal{C}(\lambda \wedge \chi) \nvdash \chi$. Hence ($\circledast$**P**) is not satisfied.

The question now is, which postulate corresponds to ($\mathcal{C}$**P**) for general semi-modular contexts? Here is the answer:

Proposition 4.5

(1) *If $\mathcal{C}$ is a semi-modular context which satisfies ($\mathcal{C}$**P**), then $\circledast_\mathcal{C}$ satisfies the following rule:*

($\circledast$**P**$'$) *If $\circledast(\lambda) \vdash \chi$ and $\circledast(\chi) \vdash \circledast(\lambda \wedge \chi)$ then $\circledast(\chi) \vdash \lambda$*

(2) *If $\circledast$ satisfies ($\circledast$**P**$'$), plus ($\circledast$**C**) and ($\circledast$**1**), then $\mathcal{C}(\circledast)$ satisfies ($\mathcal{C}$**P**).*

Proof

(1) Suppose for contradiction $\circledast_\mathcal{C}(\lambda) \vdash \chi$, $\circledast_\mathcal{C}(\chi) \vdash \circledast_\mathcal{C}(\lambda \wedge \chi)$ and $\circledast_\mathcal{C}(\chi) \nvdash \lambda$. Then from $\circledast_\mathcal{C}(\chi) \nvdash \lambda$ we know there exists $x \in [\circledast_\mathcal{C}(\chi) \wedge \neg\lambda]$. Since $\circledast_\mathcal{C}(\chi) \vdash \circledast_\mathcal{C}(\lambda \wedge \chi)$ this gives $x \in [\circledast_\mathcal{C}(\lambda \wedge \chi) \wedge \neg\lambda]$, which in turn gives $x \in [\circledast_\mathcal{C}(\lambda) \wedge \neg\lambda]$ using ($\circledast$**3**$'$). Hence $x \in \min_<([\neg\lambda])$ by Proposition 3.5. Now since $x \in [\circledast_\mathcal{C}(\chi)]$ there exists $y \in \min_<([\neg\chi])$ such that $x \preceq y$. Since $y \in \min_<([\neg\chi])$ we have $y \in [\circledast_\mathcal{C}(\chi) \wedge \neg\chi]$ by Proposition 3.5. Then, since $\circledast_\mathcal{C}(\chi) \vdash \circledast_\mathcal{C}(\lambda \wedge \chi)$ and $\neg\chi \vdash \neg(\lambda \wedge \chi)$ this gives $y \in [\circledast_\mathcal{C}(\lambda \wedge \chi) \wedge \neg(\lambda \wedge \chi)]$ and so also $y \in \min_<([\neg(\lambda \wedge \chi)])$ by Proposition 3.5. If $y \not\preceq x$ then $x < y$ by (a), contradicting the minimality of y. Hence $y \preceq x$ so $y \in [\circledast_\mathcal{C}(\lambda)]$. But $y \in [\neg\chi]$, contradicting $\circledast_\mathcal{C}(\lambda) \vdash \chi$ as required.

(2) Suppose $x \prec y$ and $y \not< x$. Then $\circledast(\neg y) \nvdash \neg x$ and $\circledast(\neg x) \vdash \neg y$. We must show $\circledast(\neg x \wedge \neg y) \vdash \neg y$, equivalently $y \nvdash \circledast(\neg x \wedge \neg y)$. But from $\circledast(\neg y) \nvdash \neg x$ and $\circledast(\neg x) \vdash \neg y$ we obtain $\circledast(\neg y) \nvdash \circledast(\neg x \wedge \neg y)$ using

($\divideontimes$**P**′). From this and ($\divideontimes$**C**) we get $\divideontimes(\neg y) \wedge y \nvdash \divideontimes(\neg x \wedge \neg y) \wedge (x \vee y)$. Since $y \vdash \divideontimes(\neg y)$ via ($\divideontimes$**1**) this is equivalent to $y \nvdash \divideontimes(\neg x \wedge \neg y) \wedge (x \vee y)$. Since obviously $y \vdash (x \vee y)$ we must have $y \nvdash \divideontimes(\neg x \wedge \neg y)$ as required.

□

It is straightforward to see ($\divideontimes$**P**′) is weaker than ($\divideontimes$**P**) given ($\divideontimes$**1**), while it implies ($\divideontimes$**P**) given ($\divideontimes$**6**).

5 Finite Base-Generated Removal

In this section we mention a concrete and important subfamily of our general family of removal functions, the ideas behind which can be seen already throughout the literature on nonmonotonic reasoning and belief change (see in particular [5] for a general treatment in a belief removal context). Given any, possibly inconsistent, set Σ of sentences, let *cons* (Σ) denote the set of all consistent subsets of Σ. We assume agent $\mathcal{A}$ is in possession of a finite set Σ of sentences which are possible *assumptions* or *defaults*, together with a strict preference ordering $\subsetplus$ on *cons* (Σ) (with sets "higher" in the ordering assumed more preferred). We assume the following two properties of $\subsetplus$:

(Σ**1**) $\subsetplus$ is a strict partial order
(Σ**2**) If $A \subset B$ then $A \subsetplus B$

(Σ**2**) is a monotonicity requirement stating a given set of defaults is strictly preferred to all its proper subsets.

Definition 5.1 If $\Sigma \subseteq L$ is a finite set of sentences and $\subsetplus$ is a binary relation over *cons* (Σ) satisfying (Σ**1**) and (Σ**2**). Then we call $\mathbb{\Sigma} = \langle \Sigma, \subsetplus \rangle$ a *prioritised default base.* If in addition $\subsetplus$ is modular then we call $\mathbb{\Sigma}$ a *modular prioritised default base.*

Lemma 5.2

(a) *If $A \supseteq B$ and $B \supsetplus C$ then $A \supsetplus C$.*
(b) *If $A \supsetplus B$ and $B \supseteq C$ then $A \supsetplus C$.*

Proof For *(a)*, if $A = B$ then it's obvious. Otherwise if $A \supset B$ then $A \supsetplus B$ by (Σ**1**) and we conclude using the transitivity of $\supsetplus$. *(b)* is proved similarly. □

In practice we might expect the ordering $\subsetplus$ over *cons* (Σ) to itself be generated from some (not necessarily total) preorder $\precsim$ over over the individual sentences in Σ (again we equate "higher" with "more preferred"). Let $E_1, \ldots, E_k$ be the equivalence classes of *cons* (Σ) under such a $\precsim$, themselves ordered in the natural way by $\precsim$, i.e., $E_1 \precsim E_2$ iff $\alpha \precsim \beta$ for some $\alpha \in E_1$ and

$\beta \in E_2$. Then to give but two prominent examples from the literature (where $\prec$ is the strict part of $\precsim$):

Inclusion-Based	[9] $A \lhd_{ib} B$ iff $\exists i$ s.t. $E_i \cap A \subset E_i \cap B$ and $\forall j$ s.t. $E_i \prec E_j$, $E_j \cap B = E_j \cap A$
Generalised-Lexicographic	[25] $A \unlhd_{gl} B$ iff $\forall i$, if $\lvert E_i \cap B\rvert < \lvert E_i \cap A\rvert$ then $\exists j$ s.t. $E_i \prec E_j$ and $\lvert E_j \cap A\rvert < \lvert E_j \cap B\rvert$. Then $\lhd_{gl}$ is the strict part of $\unlhd_{gl}$.

We remark that the inclusion-based preference usually assumes the underlying order $\precsim$ over Σ is total. For the generalised-lexicographic example, note if the preorder $\precsim$ over Σ is total then $\lhd_{gl}$ becomes modular and the generalised-lexicographic example reduces to the standard lexicographic case familiar from [3, 16].

Proposition 5.3 *Let Σ be a finite set of sentences equipped with some preorder $\precsim$ over its elements, and let $\lhd_{ib}$ and $\lhd_{gl}$ be relations over cons(Σ) defined from $\precsim$ as above. Then $\lhd_{gl}$ satisfies* $(\Sigma\mathbf{1})$ *and* $(\Sigma\mathbf{2})$, *while if $\precsim$ is total then so does $\lhd_{ib}$.*

Proof Turning first to $\lhd_{gl}$, it is obviously irreflexive, while by Proposition 1 of [25] (the first bullet point in this result) we know $\unlhd_{gl}$ is a preorder and in particular is transitive. From this it is easy to see $\lhd_{gl}$ is transitive, and so $\lhd_{gl}$ forms a strict partial order, i.e., $(\Sigma\mathbf{1})$ is satisfied. As for $(\Sigma\mathbf{2})$, suppose $A \subset B$. We know already by Proposition 1 of [25] that $A \unlhd_{gl} B$ whenever $A \subseteq B$, so it remains to show $B \not\unlhd_{gl} A$. But since $A \subset B$ we know $\lvert E_i \cap A\rvert \leq \lvert E_i \cap B\rvert$ for all i, with strict inequality for at least one i. Let $E_{i'}$ be maximal under $\prec$ such that $\lvert E_{i'} \cap A\rvert < \lvert E_{i'} \cap B\rvert$. Then $\lvert E_i \cap A\rvert \leq \lvert E_i \cap B\rvert$ for all E_i such that $E_{i'} \prec E_i$. This is enough to show $B \not\unlhd_{gl} A$.

Now turning to $\lhd_{ib}$, assume $\precsim$ is a total preorder. Then it is easy to show $(\Sigma\mathbf{1})$ and $(\Sigma\mathbf{2})$. We remark that if $\precsim$ is not total, then $\lhd_{ib}$ will be irreflexive but not transitive in general. Indeed it will not be assymetric (e.g., take Σ to consist only of two incomparable elements α, β and set $A = \{\alpha\}$ and $B = \{\beta\}$. Then both $A \lhd_{ib} B$ and $B \lhd_{ib} A$). □

How does the agent use a prioritised default base $\mathbb{\Sigma} = \langle \Sigma, \lhd \rangle$ to remove beliefs? For $\Sigma \subseteq L$ and $\lambda \in L_*$ let $cons(\Sigma, \lambda) \stackrel{\text{def}}{=} \{S \in cons(\Sigma) \mid S \nvdash \lambda\}$. Then from $\mathbb{\Sigma}$ we may define a removal function $\circledast_{\mathbb{\Sigma}}$ by setting, for each $\lambda \in L_*$,

$$\circledast_{\mathbb{\Sigma}}(\lambda) = \bigvee \left\{ \bigwedge S \mid S \in \max_{\lhd} cons(\Sigma, \lambda) \right\}.$$

In other words, after removing λ, $\mathcal{A}$ will believe precisely those sentences which are consequences of *all maximally preferred* subsets of Σ which do not imply λ.

We will now show how the family of removal functions generated from prioritised default bases fits into our general family. From a given $\mathbb{\Sigma} = \langle \Sigma, \Subset \rangle$ we may define a context $\mathcal{C}(\mathbb{\Sigma}) = (<, \prec)$ as follows. Let $sent_\Sigma(x) \stackrel{\text{def}}{=} \{\alpha \in \Sigma \mid x \in [\alpha]\}$. Then

- $x < y$ iff $sent_\Sigma(y) \Subset sent_\Sigma(x)$
- $x \prec y$ iff $sent_\Sigma(x) \nsubseteq sent_\Sigma(y)$

Thus we define x to be more plausible than y iff the set of sentences in Σ satisfied by x is more preferred than the set of sentences in Σ satisfied by y. Meanwhile we have the natural interpretation for $\prec$ that $\mathcal{A}$ has a reason to hold x to be more plausible than y precisely when one of the sentences in Σ is satisfied by x but not y.

Theorem 5.4

(1) *$\mathcal{C}(\mathbb{\Sigma})$ defined above forms a semi-modular context (which is modular if $\Subset$ is modular).*
(2) *$\preceq$ is transitive and the condition ($\mathcal{C}$**P**) from Section 4 holds.*
(3) $\circledast_{\mathbb{\Sigma}} = \circledast_{\mathcal{C}(\mathbb{\Sigma})}$.

Proof

(1) ($\mathcal{C}$**1**). First we have to show $<$ is irreflexive and transitive. This follows from the irreflexivity and transitivity of $\Subset$.
($\mathcal{C}$**2**). Irrefexivity of $\prec$, i.e., reflexivity of $\preceq$, is immediate from reflexivity of $\supseteq$.
($\mathcal{C}$**3a**). We need to show if $sent_\Sigma(x) \supseteq sent_\Sigma(y)$ and $sent_\Sigma(z) \Supset sent_\Sigma(x)$ then $sent_\Sigma(z) \Supset sent_\Sigma(y)$. This follows from Lemma 5.2(b).
(2) Transitivity of $\preceq$. Immediate from the transitivity of $\supseteq$.
($\mathcal{C}$**P**). Follows from condition (**Σ1**).
(3) Let $\lambda \in L_*$ and $\mathcal{C} = \mathcal{C}(\mathbb{\Sigma})$. We must show $[\circledast_{\mathcal{C}}(\lambda)] = \left[\circledast_{\mathbb{\Sigma}}(\lambda)\right]$.

For the left–to–right inclusion let $x \in [\circledast_{\mathcal{C}}(\lambda)]$. Then $x \preceq y$, i.e., $sent_\Sigma(x) \supseteq sent_\Sigma(y)$, for some $y \in \min_<([\neg\lambda])$. To show $x \in \left[\circledast_{\mathbb{\Sigma}}(\lambda)\right]$ it suffices to show $x \in \left[S\right]$ for some $S \in \max_{\Supset} cons(\Sigma, \lambda)$. We put $S = sent_\Sigma(y)$. The fact $x \in \left[S\right]$ follows from $sent_\Sigma(x) \supseteq sent_\Sigma(y)$. Since $y \in [\neg\lambda] \cap \left[S\right]$ we have $S \in cons(\Sigma, \lambda)$. It remains to show $T \not\Supset S$ for all $T \in cons(\Sigma, \lambda)$. But suppose it were the case that $T \in cons(\Sigma, \lambda)$ and $T \Supset S$. Since $T \nvdash \lambda$ there exists $z \in [T] \cap [\neg\lambda]$. Clearly we have $sent_\Sigma(z) \supseteq T$, so combining this with $T \Supset S$ gives $sent_\Sigma(z) \Supset S$, i.e., $sent_\Sigma(z) \Supset sent_\Sigma(y)$ by Lemma 5.2(a) and so $z < y$. But this contradicts the minimality of y. Hence $T \not\Supset S$ for all $T \in cons(\Sigma, \lambda)$ as required.

For the right–to–left inclusion let $x \in \left[\circledast_{\mathbb{\Sigma}}(\lambda)\right]$, i.e., $x \in \left[S\right]$ for some $S \in \max_{\Supset} cons(\Sigma, \lambda)$. We must find some $y \in \min_<([\neg\lambda])$ such that $sent_\Sigma(x) \supseteq sent_\Sigma(y)$. Since $S \in cons(\Sigma, \lambda)$ we know $S \nvdash \lambda$ and so there exists $y \in \left[S\right] \cap [\neg\lambda]$. Since $y \in \left[S\right]$ we know $sent_\Sigma(y) \supseteq S$. We claim in fact $sent_\Sigma(y) = S$. For if $sent_\Sigma(y) \supset S$ then $sent_\Sigma(y) \Supset S$ and then by the maximality of S we get $y \vdash \lambda$—contradiction. Hence $sent_\Sigma(y) = S$ and so, since $x \in [S]$, we know $sent_\Sigma(x) \supseteq sent_\Sigma(y)$. It remains to show $y \in \min_<([\neg\lambda])$. But if $z < y$ for some $z \in [\neg\lambda]$, i.e., $sent_\Sigma(z) \supsetneq sent_\Sigma(y)$ and so again using the maximality of $S = sent_\Sigma(y)$ we obtain $sent_\Sigma(z) \vdash \lambda$, contradicting $z \in [\neg\lambda]$. □

Thus we have shown that every removal function generated by a prioritised default base may *always* be generated by a semi-modular context which furthermore satisfies the two conditions on contexts mentioned in the previous section. By the results of the previous section, this means we automatically obtain a list of sound postulates for the default base-generated removals.

Corollary 5.5 *Let $\sum$ be any prioritised default base. Then $\circledast_\sum$ satisfies all the rules listed at the end of Section 3, as well as ($\circledast$**C**+) and ($\circledast$**P**′) from the last section.*

Note we have shown how every prioritised default base gives rise to a semi-modular context satisfying $\preceq$-transitivity and ($\mathcal{C}$**P**). An open question is whether *every* such context arises in this way.

6 AGM Preferential Removal

Recall that three of the basic AGM postulates for contraction do not hold in general for the removal functions generated by semi-modular contexts, namely Inclusion, Recovery and Vacuity. Given the influence of AGM contraction on work in this area, it is a worthwhile exercise to see if it is possible to capture all the AGM postulates in our proposed framework.

In this section we show how each of these rules can be captured. In [7] it was shown already how they may be captured within the class of modular context-generated removal. It turns out that more or less the same constructions can be used for the wider class considered here, although some complications arise regarding Vacuity.

6.1 Inclusion

The Inclusion rule is written in our setting as follows:

($\circledast$**I**) $\quad \circledast(\bot) \vdash \circledast(\lambda)$

To capture ($\circledast$**I**) for any removal generated from any semi-modular context $\mathcal{C} = (<, \prec)$, we need only to require the following condition on $\mathcal{C}$:

($\mathcal{C}$**I**) $\quad \min_<(W) \subseteq \min_\prec(W)$

According to our interpretation of $\prec$, ($\mathcal{C}$**I**) is stating that, for any world x, if $\mathcal{A}$ has some explicit reason to favour some world y over x (i.e., $y \prec x$) then in the final reckoning $\mathcal{A}$ must hold *some* world z (not necessarily the same as y) more plausible than x (i.e., $z < x$).

Proposition 6.1

(1) *If $\mathcal{C}$ satisfies* ($\mathcal{C}$**I**) *then* $\circledast_{\mathcal{C}}$ *satisfies* ($\circledast$**I**).
(2) *If* $\circledast$ *satisfies* ($\circledast$**I**) *then* $\mathcal{C}(\circledast)$ *satisfies* ($\mathcal{C}$**I**).

Proof

(1) Suppose $x \in [\circledast_{\mathcal{C}}(\bot)] = [\circledast_{\mathcal{C}}(\bot) \wedge \top]$. Then $x \in \min_<([\top]) = \min_<(W)$ by Proposition 3.5. Then by ($\mathcal{C}$**I**) for any λ we have $x \preceq y$ for all $y \in \min_<([\neg\lambda])$ and so $x \in [\circledast_{\mathcal{C}}(\lambda)]$ as required.
(2) Suppose $x \in \min_<(W)$. Then $x \in [\circledast(\bot)]$ as in part *(i)* by Proposition 3.5. By ($\circledast$**I**) we know $\circledast(\bot) \vdash \circledast(\neg y)$, so we get $x \in [\circledast(\neg y)]$, i.e., $x \preceq y$ as required. □

Given any removal function $\circledast$ we can always obtain a removal function which satisfies ($\circledast$**I**) by taking the *incarceration* $\circledast^{\mathrm{I}}$ of $\circledast$ [6]:

$$\circledast^{\mathrm{I}}(\lambda) \stackrel{\mathrm{def}}{=} \circledast(\bot) \vee \circledast(\lambda)\,.$$

Or alternatively we can modify a given context $\mathcal{C} = (<, \prec)$ into $\mathcal{C}^{\mathrm{I}} = \left(<, \prec^{\mathrm{I}}\right)$, where $x \preceq^{\mathrm{I}} y$ iff either $x \preceq y$ or $x \in \min_<(W)$. It is easy to check $\mathcal{C}^{\mathrm{I}} = \mathcal{C}\left(\circledast^{\mathrm{I}}\right)$.

6.2 Recovery

The Recovery rule is written as follows:

($\circledast$**R**) $\quad \circledast(\lambda) \wedge \lambda \vdash \circledast(\bot)$

The corresponding property on contexts $\mathcal{C} = (<, \prec)$ is:

($\mathcal{C}$**R**) $\quad$ If $y \notin \min_<(W)$ and $x \neq y$ then $x \prec y$

Thus the only worlds $\nabla_{\preceq}(x)$ contains, other than x itself, are worlds in $\min_<(W)$.

Proposition 6.2

(1) *If $\mathcal{C}$ satisfies* ($\mathcal{C}$**R**) *then* $\circledast_{\mathcal{C}}$ *satisfies* ($\circledast$**R**).
(2) *If* $\circledast$ *satisfies* ($\circledast$**R**) *then* $\mathcal{C}(\circledast)$ *satisfies* ($\mathcal{C}$**R**).

Proof

(1) Suppose $x \in [\circledast_{\mathcal{C}}(\lambda) \wedge \lambda]$. Then $x \preceq y$ for some $y \in \min_<([\neg\lambda])$. If $x = y$ then $x \in [\neg\lambda]$—contradiction. Hence by ($\mathcal{C}$**R**) we get $x \in \min_<(W)$, i.e., $x \in [\circledast_{\mathcal{C}}(\bot)]$.
(2) Suppose $x \preceq y$, i.e., $x \in [\circledast(\neg y)]$. If $x = y$ we are done so assume $x \neq y$. Then $x \in [\neg y]$. Then from $x \in [\circledast(\neg y) \wedge \neg y]$ we get $x \in [\circledast(\bot)]$ by ($\circledast$**R**). This is equivalent to the desired $x \in \min_<(W)$. □

Note the combination of ($\mathcal{C}$**I**) and ($\mathcal{C}$**R**) specifies $\prec$, equivalently $\preceq$, uniquely in terms of $<$, viz. $x \preceq_{agm} y$ iff $x = y$ or $x \in \min_< (W)$, and we obtain the removal recipe of AGM contraction, in which removal of λ boils down to just adding the $<$-minimal $\neg\lambda$-worlds to the $<$-minimal worlds:

$$[\circledast_{agm}(\lambda)] = \min_< (W) \cup \min_< ([\neg\lambda]) .$$

It is easy to check that the resulting context $\mathcal{C}$ satisfies condition ($\mathcal{C}$**3a**) and thus forms a semi-modular context. It is also easy to check ($\mathcal{C}$**P**) is satisfied and that the above-defined $\preceq_{agm}$ is transitive. Thus the above $\circledast_{agm}$ also satisfies ($\circledast$**C**+) and ($\circledast$**P**′) from Section 4. It can also be shown to satisfy ($\circledast$**6b**).

6.3 Vacuity

The Vacuity rule is written as follows:

($\circledast$**V**) If $\circledast(\bot) \nvdash \lambda$ then $\circledast(\lambda) \equiv \circledast(\bot)$

Unlike in the modular case, where Vacuity is known to follow from Inclusion for modular removal functions [7], ($\circledast$**V**) does not even hold in general for the above preferential AGM contraction $\circledast_{agm}$. This was essentially noticed, in a revision context, in [4].

Example 6.3 Let $L = \{p, q\}$ and $<= \{(11, 01)\}$. So $\left[\circledast_{agm}(\bot)\right] = \{00, 11, 10\}$. Let $\lambda = p$. Then we have $\circledast_{agm}(\bot) \nvdash \lambda$ (because $00 \in \left[\circledast_{agm}(\bot)\right]$), but $\min_<$ $([\neg\lambda]) = \{00, 01\}$, so $\left[\circledast_{agm}(\lambda)\right] = \min_< (W) \cup \min_< ([\neg\lambda]) = W \neq \left[\circledast_{agm}(\bot)\right]$.

In order to ensure $\circledast_{agm}$ satisfies ($\circledast$**V**) it is necessary, as is done in [14], to enforce the following property on $<$.

($<$ **V**) $\forall x, y\, ((x \in \min_< (W) \wedge y \notin \min_< (W)) \rightarrow x < y)$.

In other words all $<$-minimal worlds can be compared with, and are below, every world which is not $<$-minimal. For general semi-modular contexts $\mathcal{C} = (<, \prec)$ we also require the following condition, which is weaker than ($\mathcal{C}$**I**):

($\mathcal{C}$**V**) If $x, y \in \min_<(W)$ then $x \nprec y$

This property says that for any two of his $<$-minimal worlds, $\mathcal{A}$ will not have explicit reason to hold one to be more plausible than the other.

For basic removal [7] the following direction of Vacuity is always valid:

($\circledast$**V**$_1$) If $\circledast(\bot) \nvdash \lambda$ then $\circledast(\lambda) \vdash \circledast(\bot)$

Proposition 6.4 ($\circledast$**V**$_1$) *is not valid for* $\circledast_{\mathcal{C}}$*, even if* $\mathcal{C}$ *satisfies* ($\mathcal{C}$**V**).

Proof For the counterexample, assume $L = \{p, q\}$ and let $<= \{(01, 11), (10, 00)\}$ and $\preceq = \{(01, 10), (10, 01)\}$ (plus reflexivity). Then $[\circledast_{\mathcal{C}}(\bot)] = \{01, 10\}$, so $\mathcal{C}$ clearly satisfies ($\mathcal{C}$**V**). Let $\lambda = p$. Clearly $\circledast_{\mathcal{C}}(\bot) \nvdash p$, while $[\circledast_{\mathcal{C}}(p)] = \{01, 10, 00\} \nsubseteq [\circledast_{\mathcal{C}}(\bot)]$. □

But if we insist on $(< \mathbf{V})$ as well as $(\mathcal{C}\mathbf{V})$ then we get it:

Proposition 6.5

(1) *If* $\mathcal{C}$ *satisfies* $(\mathcal{C}\mathbf{V})$ *and* $(< \mathbf{V})$ *then* $\circledast_{\mathcal{C}}$ *satisfies* $(\circledast\mathbf{V})$.
(2) *If* $\circledast$ *satisfies* $(\circledast\mathbf{V})$ *then* $\mathcal{C}(\circledast)$ *satisfies* $(\mathcal{C}\mathbf{V})$.

Proof

(1) Suppose $\circledast_{\mathcal{C}}(\bot) \nvdash \lambda$. Then there exists $x_0 \in [\circledast_{\mathcal{C}}(\bot) \wedge \neg\lambda]$. By $(\circledast\mathbf{3}')$ we know $\circledast_{\mathcal{C}}(\bot) \wedge \neg\lambda \vdash \circledast_{\mathcal{C}}(\lambda)$ so $x_0 \in [\circledast_{\mathcal{C}}(\lambda) \wedge \neg\lambda]$.
To show $\circledast_{\mathcal{C}}(\bot) \vdash \circledast_{\mathcal{C}}(\lambda)$ suppose $y \in [\circledast_{\mathcal{C}}(\bot)]$. Then $y \in \min_<(W)$. Since also $x_0 \in [\circledast_{\mathcal{C}}(\bot)]$ we know also $x_0 \in \min_<(W)$ and so $y \preceq x_0$ by $(\mathcal{C}\mathbf{V})$. But from $x_0 \in [\circledast_{\mathcal{C}}(\lambda) \wedge \neg\lambda]$ we get $x_0 \in \min_<([\neg\lambda])$ by Proposition 3.5 which gives $y \in [\circledast_{\mathcal{C}}(\lambda)]$ as required.
To show $\circledast_{\mathcal{C}}(\lambda) \vdash \circledast_{\mathcal{C}}(\bot)$ it suffices by $(\circledast\mathbf{C})$ to show $\circledast_{\mathcal{C}}(\lambda) \wedge \neg\lambda \vdash \circledast_{\mathcal{C}}(\bot)$. So let $y \in [\circledast_{\mathcal{C}}(\lambda) \wedge \neg\lambda]$. Then $y \in \min_<([\neg\lambda])$ by Proposition 3.5. We must show $y \in [\circledast_{\mathcal{C}}(\bot)]$, i.e., $y \in \min_<(W)$. But if $y \notin \min_<(W)$ then $x_0 < y$ using $(< \mathbf{V})$. But since $x_0 \in [\neg\lambda]$ this contradicts $y \in \min_<([\neg\lambda])$. Hence $y \in \min_<(W)$ as required.
(2) If $x \in \min_<(W)$ then $x \vdash \circledast(\bot)$ and so $\circledast(\bot) \nvdash \neg x$. Then appling $(\circledast\mathbf{V})$ to this gives $\circledast(\neg x) \equiv \circledast(\bot)$. So if both $x, y \in \min_<(W)$ then $\circledast(\neg x) \equiv \circledast(\bot) \equiv \circledast(\neg y)$. Hence $\circledast(\neg y) \nvdash \neg x$ by $(\circledast\mathbf{1})$, i.e., $x \preceq y$ as required to show $(\mathcal{C}\mathbf{V})$. □

7 Conclusion

In this paper we introduced a family of removal functions, generalising the one given in [7] to allow for incomparabilities in the plausibility relation $<$ between possible worlds. Removal is carried out using the plausibility relation in combination with a second relation $\prec$ which can be thought of as indicating "reasons" for holding one world to be more plausible than another. We axiomatically characterised this general family as well as certain subclasses, and we showed how this family includes some important and natural families of belief removal, specifically those which may be generated from prioritised default bases and the preferential counterpart of AGM contraction. Our results show the central construct used in this paper, i.e., semi-modular contexts, to be a very useful tool in the study of belief removal functions.

For future work we would like to locate further subclasses of interest, for example the counterparts in this setting of systematic withdrawal [19] and severe withdrawal [23]. We would also like to employ semi-modular contexts in the setting of *social belief removal* [8], in which there are several agents, each assumed to have their own removal function, and in which all agents must remove some belief to become consistent with each other. Booth and Meyer [8] showed that, under the assumption that each agent uses a removal function generated from a *modular* context, certain *equilibrium points* in the social removal process are guaranteed to exist. An interesting question would be whether these results generalise to the *semi-modular* case. Since semi-modular contexts are built from strict partial orders, this question should also be of some relevance to the problem of *aggregating strict partial orders* [20].

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