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On Revising Fuzzy Belief Bases

Abstract. We look at the problem of revising fuzzy belief bases, i.e., belief base revision in which both formulas in the base as well as revision-input formulas can come attached with varying degrees. Working within a very general framework for fuzzy logic which is able to capture certain types of uncertainty calculi as well as truth-functional fuzzy logics, we show how the idea of rational change from "crisp" base revision, as embodied by the idea of partial meet (base) revision, can be faithfully extended to revising fuzzy belief bases. We present and axiomatise an operation of partial meet *fuzzy base* revision and illustrate how the operation works in several important special instances of the framework. We also axiomatise the related operation of partial meet fuzzy base *contraction*.[1]

Keywords: Belief revision, base revision, partial meet, fuzzy logic.

1. Introduction

The ability to rationally change one's beliefs in the face of new information which, possibly, contradicts the currently held beliefs is a basic characteristic of intelligent behaviour. Hence the question of belief revision is an important question in philosophy and Artificial Intelligence. A very successful framework in which this question is studied is the one due to Alchourrón, Gärdenfors and Makinson (AGM) [1, 7], with its operation of *partial meet revision*. One limitation of this framework is that belief in a formula is taken as a matter of all or nothing: either the formula is believed or it is not. However, real-life knowledge bases may well contain information of a more *graded* nature. For instance we might want to represent information about *vague* concepts or *uncertain* beliefs. Likewise revision inputs may come with a degree attached. Our aim in this paper is to examine revision in the general setting which allows for such different degrees, while keeping the spirit of AGM.

As a most suitable backdrop in which to work out our ideas we choose a very general framework for *fuzzy logic* due to Gerla [10]. The basic construct here is that of an *abstract fuzzy deduction system*, which generalises Tarski's notion of deductive systems. Roughly, this consists of three basic ingredients: *(i)* a set L of formulas to describe the world, *(ii)* a set W of *degrees*

[1] This paper is an extended version of a paper presented at the Nineteenth Conference on Uncertainty in Artificial Intelligence (UAI'03).

Presented by **Jacek Malinowski**; *Received* March 22, 2004

Studia Logica (2005) 80: 29–61
DOI: 10.1007/s11225-005-6775-5

(whose precise interpretation is mostly left open) which may be assigned to the formulas to create *fuzzy belief bases*, and *(iii)* a *fuzzy deduction operator* D which takes as input a fuzzy base u and returns another fuzzy base $D(u)$ representing its *(fuzzy) conclusions*. Sometimes a fourth ingredient is included – a *fuzzy semantics* $\mathcal{M}$ – in which case we speak of an *abstract fuzzy logic*. When $W = \{0, 1\}$ we find ourselves in the usual "crisp" setting of AGM. The framework has also been shown capable of capturing several different flavours of uncertain reasoning, including truth-functional logic and certain types of probabilistic logic.

Within this fuzzy framework, the question of revision we are interested in then takes the following form: Given a fuzzy base u representing our current information, how should we change u to incorporate the new information that the degree of some formula φ is at least a for some $a \in W$? In this paper we assume that the object of change u is an arbitrary fuzzy base which need not be deductively closed, i.e., possibly $u \neq D(u)$. That is, we differentiate between those beliefs which are "basic" or "explicit", (u) and those which are "merely derived" or "implicit" (i.e., that information in $D(u)$ which goes strictly beyond that contained in u).

The original AGM theory was a theory about how to revise deductively closed sets of formulas, but the more general case of revising arbitrary (crisp) bases has also been studied, notably by Hansson [15, 17], who axiomatically characterised partial meet *base* revision. We will generalise this operation into *partial meet fuzzy base revision* and give an axiomatisation. Surprisingly, despite the increase in complexity which admitting many degrees brings, the form of the axiomatisation is roughly the same as in the crisp case. This shows how the principles on which partial meet revision are based really require very little structure. The set of degrees is not even required to be linearly-ordered – any complete, distributive lattice will do.

The plan of the paper is as follows. In Section 2 we set up the framework of abstract fuzzy logic and describe some instances of it, including those related to truth-functional fuzzy logics, necessity logic and probability logic. In Section 3 we define partial meet fuzzy base revision operators and give examples to illustrate how these operators work for each instance of the framework from the previous section. We give the axiomatisation of partial meet fuzzy base revision in Section 4. In Section 5 we consider and axiomatise the two special limiting cases of partial meet fuzzy base revision – full meet and maxichoice fuzzy base revision. Then, in Section 6 we axiomatise the related operation of partial meet fuzzy base *contraction*. In Section 7 we briefly reflect on the generality of our results before concluding in Section 8. An Appendix recapitulates a proof system given in [8, 9] for one of our examples of abstract fuzzy deduction system, namely the one related to probability logic.

2. Abstract fuzzy logic

Our first task is to formally define abstract fuzzy deduction systems. The following definitions are based on [10]. As we said above, we assume L to be the set of all formulas. We take the set W of all possible degrees to be a complete lattice, i.e., we assume W to come equipped with a partial order $\leq_W$ on W such that every $A \subseteq W$ has both a *supremum* (or *join*) $\sup(A)$ and an *infimum* (or *meet*) $\inf(A)$. For $a, b \in W$ we write $a \curlyvee b$ for $\sup(\{a, b\})$ and $a \curlywedge b$ for $\inf(\{a, b\})$. Often (for instance in our examples) W will be linearly ordered (e.g., the real unit interval). However, in general the only additional assumption we make about W is that it is also *distributive*, i.e., that for all $a, b, c \in W$ we have $a \curlywedge (b \curlyvee c) = (a \curlywedge b) \curlyvee (a \curlywedge c)$, equivalently, $a \curlyvee (b \curlywedge c) = (a \curlyvee b) \curlywedge (a \curlyvee c)$.[2] We use 0_W and 1_W to denote the minimal and maximal elements of W. (For an introduction to lattice theory we refer the reader to [3].)

A *fuzzy belief base* is then just an assignment $u : L \to W$ of degrees to the formulas. Such a piece of information u should be understood as an under constraint, i.e., $u(\varphi) = a$ means that the degree of φ is *at least* a. We denote the set of all possible fuzzy bases by $\mathcal{F}(L)$. The ordering $\leq_W$ induces a "fuzzy subset" relation $\sqsubseteq$ on $\mathcal{F}(L)$ by taking, for $u, v \in \mathcal{F}(L)$, $u \sqsubseteq v$ iff $u(\varphi) \leq_W v(\varphi)$ for all $\varphi \in L$. The meaning of this is that v carries more (or more exact) information than u. With this definition it is easy to see that $(\mathcal{F}(L), \sqsubseteq)$ forms a complete, distributive lattice. Given $X \subseteq \mathcal{F}(L)$ we shall denote the supremum and infimum of X under $\sqsubseteq$ by $\bigsqcup X$ and $\bigsqcap X$ respectively[3]. We write $u \sqcup v$ for $\bigsqcup\{u, v\}$ and $u \sqcap v$ for $\bigsqcap\{u, v\}$. We have the following, for all $X \subseteq \mathcal{F}(L)$ and $\varphi \in L$,

$$\left[\bigsqcup X\right](\varphi) = \sup(\{u(\varphi) \mid u \in X\})$$

$$\left[\bigsqcap X\right](\varphi) = \inf(\{u(\varphi) \mid u \in X\}).$$

We use $\sqsubset$ to denote the strict part of $\sqsubseteq$. The $\sqsubseteq$-maximal element of $\mathcal{F}(L)$, i.e., the fuzzy base which assigns degree 1_W to every formula, will be denoted

[2]For another general approach to modelling uncertainty which likewise relaxes the assumption of linearity see [13].

[3]Note in general one should not assume $\bigsqcup X \in X$ or $\bigsqcap X \in X$.

by $u_\perp$. The $\sqsubseteq$-minimal element of $\mathcal{F}(L)$, i.e., the fuzzy base which assigns degree 0_W to every formula, will be denoted by $u_\top$. For a fuzzy base u we call the set of formulas φ for which $u(\varphi) \neq 0_W$ the *support of* u and denote this set by $Supp(u)$. If $Supp(u) = \{\varphi_1, \ldots, \varphi_k\}$ is finite then we may represent u as $\{(\varphi_1/a_1), \ldots, (\varphi_k/a_k)\}$ with the interpretation that $u(\varphi_i) = a_i$ for $i = 1, \ldots, k$. We will often use (φ/a) to denote the base $\{(\varphi/a)\}$. Thus $(\varphi/a) \sqsubseteq u$ will sometimes serve as alternative notation for $a \leq_W u(\phi)$. Although the support of a fuzzy base will typically be finite, the results we describe will be valid for arbitrary u.

The tool for drawing conclusions is the fuzzy deduction operator $D : \mathcal{F}(L) \to \mathcal{F}(L)$. It is assumed to satisfy analogues of the three basic Tarski properties:

- $u \sqsubseteq D(u)$ **(Reflexivity)**
- $u \sqsubseteq v$ implies $D(u) \sqsubseteq D(v)$ **(Monotony)**
- $D(D(u)) = D(u)$ **(Idempotence)**

If $D(u) = u_\perp$ then we say that u is *D-inconsistent*, otherwise *D-consistent*. (We omit the "*D*-" if it is clear from the context.) A consequence of Monotony which will be relied upon repeatedly in our proofs later on is the following, which expresses that the set of D-consistent fuzzy bases is *downwards closed* in the lattice $\mathcal{F}(L)$:

- If v is D-consistent and $u \sqsubseteq v$ then u is D-consistent **(Con↓)**

A *(fuzzy) theory* is any fixed point of D. Another property of D, which will be important to us, is *logical compactness*:

Definition 2.1 ([10]). Let $D : \mathcal{F}(L) \to \mathcal{F}(L)$ be a deduction operator. Then D is *logically compact* iff we have $D(\bigsqcup X) \neq u_\perp$ for all $X \subseteq \mathcal{F}(L)$ such that *(i)* $u \in X$ implies $D(u) \neq u_\perp$, and *(ii)* for all $u, v \in X$ there exists $w \in X$ such that $u \sqcup v \sqsubseteq w$.

Using an order-theoretical term, the definition says that D is logically compact iff the supremum of every *directed* family of D-consistent fuzzy bases is itself D-consistent.

We are now able to give the following formal definition:

Definition 2.2. An *abstract fuzzy deduction system* is a triple (L, W, D) where L is a set of formulas, W is a complete, distributive lattice of degrees and D is a logically compact fuzzy deduction operator which satisfies Monotony, Idempotence and Reflexivity.

Sometimes (especially for our examples) it is convenient to describe the deduction operator D of an abstract fuzzy deduction system semantically. An *abstract fuzzy semantics* is a subset $\mathcal{M}$ of $\mathcal{F}(L)$, such that $u_\perp \notin \mathcal{M}$, whose elements are called *models*. For now it does not hurt for the reader to think of the models as complete descriptions of "possible worlds", whereas the fuzzy bases u not in $\mathcal{M}$ represent incomplete knowledge. However, it is important to note that in fact *any* set of fuzzy bases will qualify as a valid semantics provided it does not contain $u_\perp$. An element $m \in \mathcal{M}$ is a model of a fuzzy base u if $u \sqsubseteq m$. We denote the set of models of u in $\mathcal{M}$ by $\mathrm{mod}_\mathcal{M}(u)$. An abstract fuzzy semantics $\mathcal{M}$ yields a fuzzy deduction operator $J_\mathcal{M}$ by setting, for each $u \in \mathcal{F}(L)$,

$$J_\mathcal{M}(u) = \bigsqcap \mathrm{mod}_\mathcal{M}(u).$$

It is easy to see that $J_\mathcal{M}$ satisfies Monotony, Idempotence and Reflexivity, and also that a fuzzy base u is $J_\mathcal{M}$-consistent iff $\mathrm{mod}_\mathcal{M}(u) \neq \emptyset$.

Definition 2.3. An *abstract fuzzy logic* is a quadruple $(L, W, D, \mathcal{M})$ where (L, W, D) is an abstract fuzzy deduction system and $\mathcal{M}$ is an abstract fuzzy semantics such that $D = J_\mathcal{M}$ (i.e., the "completeness theorem" holds).

For any abstract fuzzy deduction system we can always associate a suitable semantics: just take $\mathcal{M}$ to be the set of all D-consistent theories.

2.1. Concrete examples

We now give a few example instantiations of the above framework. In each of these we take the set of formulas to be the set of formulas L_{Prop} from a propositional language closed under the connectives $\neg, \wedge, \vee$ and $\rightarrow$. We treat $\theta \leftrightarrow \varphi$ as an abbreviation for $(\theta \rightarrow \varphi) \wedge (\varphi \rightarrow \theta)$. We denote the classical logical consequence operator of propositional logic by Cn.

2.1.1. Crisp deduction systems

The simplest example of a set of degrees is, of course, the case when W consists of just two elements $\{0, 1\}$ standing for "false" and "true" respectively. In this case belief bases u are "crisp", i.e., they correspond to (characteristic functions of) sets of formulas in L_{Prop}, and $\sqsubseteq, \sqcap, \sqcup$ effectively reduce to the usual $\subseteq, \cap, \cup$ (thus in this case we write the more usual "$\varphi \in u$" rather than "$u(\varphi) = 1$" etc.). In the belief revision literature it is customary to assume that, in addition to Monotony, Idempotence and Reflexivity, the deduction operator D satisfies the following three rules:

- If $\varphi \in Cn(u)$ then $\varphi \in D(u)$ (Supraclassicality)
- $\varphi \in D(u \cup \{\theta\})$ iff $(\theta \to \varphi) \in D(u)$ (Deduction)
- If $\varphi \in D(u)$ then $\varphi \in D(u')$ for some finite $u' \subseteq u$ (Compactness)

We will call an abstract fuzzy deduction system of the form $(L_{\text{Prop}}, \{0,1\}, D)$ where D satisfies the above three properties a *crisp deduction system.* That D is logically compact follows from the following observation:

PROPOSITION 2.4. *Let* $D : 2^{L_{\text{Prop}}} \to 2^{L_{\text{Prop}}}$ *be a deduction operator which satisfies* Supraclassicality *and* Deduction*. Then D satisfies* Compactness *iff D is logically compact in the sense of Definition 2.1.*

PROOF. To show D is logically compact given that D satisfies Compactness, let X be a set of bases satisfying conditions *(i)* and *(ii)* from Definition 2.1, i.e., *(i)* $u \in X$ implies u is D-consistent, and *(ii)* $u, v \in X$ implies there exists $w \in X$ such that $u \cup v \subseteq w$. We must show $\bigcup X$ is D-consistent. But suppose for contradiction that $\bigcup X$ is D-inconsistent. Then, for all $\theta \in L_{\text{Prop}}$ we have $\theta \in D(\bigcup X)$. In particular $\perp \in D(\bigcup X)$, where $\perp$ denotes any fixed classical contradiction. By Compactness we know that $\perp \in D(u')$ for some finite $u' \subseteq \bigcup X$. Suppose $u' = \{\varphi_1, \ldots, \varphi_n\}$. For each φ_i choose $u_i \in X$ such that $\varphi_i \in u_i$. Then $u' \subseteq u_1 \cup \ldots \cup u_n$. So $D(u') \subseteq D(u_1 \cup \ldots \cup u_n)$ by Monotony. Hence we have constructed some finite set $\{u_1, \ldots, u_n\} \subseteq X$ such that $\perp \in D(u_1 \cup \ldots \cup u_n)$. From repeated use of condition *(ii)* above we know there exists $w \in X$ such that $u_1 \cup \ldots \cup u_n \subseteq w$. But for any such w we have $\perp \in D(w)$ by Monotony and so, from classical logic, $\theta \in Cn(D(w))$ for all $\theta \in L_{\text{Prop}}$. Now from Supraclassicality and Idempotence it can be shown that $Cn(D(w)) \subseteq D(w)$, hence we obtain $\theta \in D(w)$ for all $\theta \in L_{\text{Prop}}$ and so w is D-inconsistent. But from $w \in X$ we know already that w is D-consistent. This gives the required contradiction and so we must have $\bigcup X$ is D-consistent as desired.

To show D satisfies Compactness if D is logically compact, let u be a base and let $\varphi \in L_{\text{Prop}}$ be such that, for all finite subsets $u' \subseteq u$, we have $\varphi \notin D(u')$. We must show that this implies $\varphi \notin D(u)$. But consider the set of bases $X = \{u' \cup \{\neg\varphi\} \mid u' \text{ finite and } u' \subseteq u\}$. Now, for each finite $u' \subseteq u$, using Deduction along with the fact that $Cn(D(u')) \subseteq D(u')$ allows us to deduce from $\varphi \notin D(u')$ that $u' \cup \{\neg\varphi\}$ is D-consistent. Meanwhile it is easy to see that $u_1, u_2 \in X$ implies $u_1 \cup u_2 \in X$. Hence the set X satisfies conditions *(i)* and *(ii)* from Definition 2.1 and so, since D is logically compact, we deduce $\bigcup X = u \cup \{\neg\varphi\}$ is D-consistent. From this, using Monotony, Reflexivity and the fact that $Cn(D(u \cup \{\neg\varphi\})) \subseteq D(u \cup \{\neg\varphi\})$, we get $\varphi \notin D(u)$ as required. ■

Thus we see that, for crisp deduction systems, the property of logical compactness collapses into the usual notion of compactness. Note that for a semantics here we could take $\mathcal{M}$ to consist of all the maximal consistent theories.

2.1.2. Łukasiewicz fuzzy logic

In the rest of our examples we take $W = [0,1]$, i.e., the real unit interval equipped with the usual ordering $\leq$. Each example will differ only in the choice of a semantics, i.e., what counts as a "possible world". This leads to different types of deduction operator as well as different interpretations of what the degrees stand for. The first example is related to infinitely many-valued Łukasiewicz logic (see, for example [11, 20]). We take as the semantics the set $\mathcal{M}_{\mathrm{luk}}$ of all *truth-functional valuations* over L_{Prop} in the many-valued Łukasiewicz logic, i.e., the set of functions $m : L_{\mathrm{Prop}} \to [0,1]$ satisfying, for all $\theta, \varphi \in L_{\mathrm{Prop}}$,

$$\begin{array}{rcl} m(\neg\theta) & = & 1 - m(\theta) \\ m(\theta \wedge \varphi) & = & m(\theta) \barwedge m(\varphi) \\ m(\theta \vee \varphi) & = & m(\theta) \veebar m(\varphi) \\ m(\theta \to \varphi) & = & 1 \barwedge (1 - m(\theta) + m(\varphi)) \end{array}$$

(Note that here "$\to$" does not behave as material implication.) So here the "fuzziness" arises from having worlds with graded properties, i.e., the degrees are interpreted as degrees of truth. We then take $D_{\mathrm{luk}} = J_{\mathcal{M}_{\mathrm{luk}}}$. It can be shown [20, Lemma 4.17] that for any given fuzzy base u we have

$$u \sqcup (\varphi/a) \text{ is inconsistent iff } D_{\mathrm{luk}}(u)(\neg\varphi) > 1 - a \qquad (*)$$

We also have the following:

PROPOSITION 2.5 ([10]). *D_{luk} is logically compact.*

For an example of a fuzzy base in this logic let x, y, z be distinct propositional variables and consider:

$$u_0 = \{(x/0.75), (x \to y/0.75), (z/0.25)\}.$$

For an example of an inference we have $D_{\mathrm{luk}}(u_0)(y) = 0.5$, i.e., we infer from u_0 that the truth-degree of y is at least 0.5. To see this, we have

$$D_{\mathrm{luk}}(u_0)(y) = \inf\{m(y) \mid m \in \mathrm{mod}_{\mathcal{M}_{\mathrm{luk}}}(u_0)\}.$$

Hence it suffices to show that $0.5 \leq m(y)$ for all $m \in \mathrm{mod}_{\mathcal{M}_{\mathrm{luk}}}(u_0)$, with equality holding for at least one m. So let $m \in \mathrm{mod}_{\mathcal{M}_{\mathrm{luk}}}(u_0)$. Then we have $0.75 \leq m(x)$, $0.75 \leq m(x \to y)$, and $0.25 \leq m(z)$. Unpacking the second constraint gives us $0.75 \leq 1 \curlywedge (1 - m(x) + m(y))$ which leads to $m(x) - 0.25 \leq m(y)$. Since $0.75 \leq m(x)$ this gives us the desired $0.5 \leq m(y)$. Furthermore, we can obtain equality here by choosing $m_0 \in \mathrm{mod}_{\mathcal{M}_{\mathrm{luk}}}(u_0)$ such that $m_0(x) = 0.75, m_0(y) = 0.5$ and $m_0(z) = 0.25$. Hence $D_{\mathrm{luk}}(u_0)(y) = 0.5$ as required. By similar reasoning we can also show $D_{\mathrm{luk}}(u_0)(y \wedge z) = \min\{0.5, 0.25\} = 0.25$, i.e., we infer that the truth-degree of $y \wedge z$ is at least 0.25. So, by $(*)$ above, we know $u_0 \sqcup (\neg(y \wedge z)/b)$ will be inconsistent for any $b > 0.75$.

2.1.3. Necessity logic

Our final two examples show how the framework is also able to capture some types of non-truth-functional belief. The first of these, which corresponds to possibilistic logic [6], was described within this framework in [9]. For the semantics we take the set $\mathcal{M}_{\mathrm{N}}$ of all *necessity functions* over L_{Prop}, i.e., the set of functions $n : L_{\mathrm{Prop}} \to [0, 1]$ which satisfy, for all $\theta, \varphi \in L_{\mathrm{Prop}}$,

(N1) If $\theta \in Cn(\emptyset)$ then $n(\theta) = 1$ and $n(\neg\theta) = 0$.

(N2) If $(\theta \leftrightarrow \varphi) \in Cn(\emptyset)$ then $n(\theta) = n(\varphi)$.

(N3) $n(\theta \wedge \varphi) = n(\theta) \curlywedge n(\varphi)$.

The degrees become now degrees of necessity. We then take $D_{\mathrm{N}} = J_{\mathcal{M}_{\mathrm{N}}}$.

PROPOSITION 2.6 ([9]). *D_{N} is logically compact.*

As is shown in [9], in this logic the notion of consistency is reducible to classical propositional consistency, in that a fuzzy base u is D_{N}-consistent iff $Supp(u)$ is Cn-consistent. Also, if u is consistent (and $\varphi \notin Cn(\emptyset)$) then $D_{\mathrm{N}}(u)(\varphi)$ may be determined from the values given to those formulas which classically imply φ as follows:

$$D_{\mathrm{N}}(u)(\varphi) = \sup\{u(\theta_1) \curlywedge \ldots \curlywedge u(\theta_k) \mid \varphi \in Cn(\{\theta_1, \ldots, \theta_k\})\}$$

(If $\varphi \in Cn(\emptyset)$ then clearly $D_{\mathrm{N}}(u)(\varphi) = 1$.) For example using the same fuzzy base u_0 as in the previous example we get $D_{\mathrm{N}}(u_0)(y) = 0.75$ and $D_{\mathrm{N}}(u_0)(y \wedge z) = 0.25$.

2.1.4. Probability logic (lower envelopes)

Our last example is probabilistic. It is the logic of "lower envelopes" studied in [8].[4] This time we take as a semantics the set $\mathcal{M}_{\rm P}$ of all *probability functions* over $L_{\rm Prop}$, i.e., all functions $p : L_{\rm Prop} \to [0,1]$ which satisfy, for all $\theta, \varphi \in L_{\rm Prop}$,

(P1) If $\theta \in Cn(\emptyset)$ then $p(\theta) = 1$.

(P2) If $\neg(\theta \wedge \varphi) \in Cn(\emptyset)$ then $p(\theta \vee \varphi) = p(\theta) + p(\varphi)$.

Then every "world" contains complete information of a random phenomena. We then take $D_{\rm P} = J_{\mathcal{M}_{\rm P}}$.

PROPOSITION 2.7 ([8, 9]). *$D_{\rm P}$ is logically compact.*

A fuzzy base u then gives a lower constraint for an unknown probability distribution. The deduction operator $D_{\rm P}(u)$ improves the initial constraint. It is easy to see (using **(P2)**) that $D_{\rm P}$ satisfies the property $(*)$ mentioned in Section 2.1.2. A syntactic characterisation of $D_{\rm P}$ is given in [9] and is reproduced in the appendix for the interested reader. For an example of an inference in this logic consider again the fuzzy base u_0 from the previous examples. Then $D_{\rm P}(u_0)(y) = 0.5$, i.e., we infer that the probability of y is at least 0.5. To see this, we have

$$D_{\rm P}(u_0)(y) = \inf\{p(y) \mid p \in \mathrm{mod}_{\mathcal{M}_{\rm P}}(u_0)\}.$$

Hence it suffices to show $0.5 \leq p(y)$ for all $p \in \mathrm{mod}_{\mathcal{M}_{\rm P}}(u_0)$, with equality holding for at least one p. To see this, first note that, using the properties of probability functions, we get $p(x) = p(x \wedge y) + p(x \wedge \neg y)$ and $p(x \to y) = 1 - p(\neg(x \to y)) = 1 - p(x \wedge \neg y)$. Hence we may rewrite the first two constraints on p as

$$0.75 \leq p(x \wedge y) + p(x \wedge \neg y) \text{ and } p(x \wedge \neg y) \leq 0.25.$$

The first constraint gives $0.75 - p(x \wedge \neg y) \leq p(x \wedge y)$. Then using this with the second constraint gives $0.5 \leq p(x \wedge y)$. Since $p(x \wedge y) \leq p(y)$ for any probability function we then get $0.5 \leq p(y)$ as required. We obtain equality by choosing any $p_0 \in \mathrm{mod}_{\mathcal{M}_{\rm P}}(u_0)$ such that $p_0(x \wedge \neg y) = p_0(\neg x \wedge \neg y) = 0.25$ and $p_0(x \wedge y) = 0.5$.

Note here the answer for $D_{\rm P}(u_0)(y)$ coincides with that for $D_{\rm luk}(u_0)(y)$ in the Łukasiewicz example above. In general, though, the two deduction

[4] See also [9] for some more examples of "probability-like" logics within this framework.

operators will give different results.[5] For example it can be shown that, in contrast to $D_{\text{luk}}(u_0)$, we get $D_{\text{P}}(u_0)(y \wedge z) = 0$.

3. Fuzzy base revision

Now we have set up the basic framework we can state formally the question of revision we are interested in:

> **Question.** Assume a fixed abstract fuzzy deduction system (L, W, D) as background. Then given a fuzzy belief base u (representing our current (fuzzy) information) and a pair $(\varphi/a) \in L \times W$ (representing the new information that the degree of φ is at least a), how should we determine $u \star (\varphi/a)$ which represents the *revision* of u to consistently incorporate the new information (φ/a)?

The special case of crisp deduction systems is the case which is considered in the AGM framework. The idea there is to decompose the operation into two main steps. *First*, the initial (crisp) base u is altered if necessary so as to "make room" for, i.e., become consistent with, the incoming crisp formula φ. This is achieved by making u deductively weaker (contraction). Here we should adhere to the principle of *minimal change*, according to which this weakening should be made as "small" as possible. (See [22] for a discussion of this principle.) *Then* the new formula is simply joined on to the result (expansion)[6]. In *partial meet revision* [1, 15] the idea is to focus for the first step on those subsets of u which are consistent with φ and which are *maximal* with this property. Then, a certain number of the elements of this set are somehow selected as the "best" or "most preferred" and then their intersection is taken. The result of this intersection is then expanded by φ. We would like to generalise this procedure to apply to an arbitrary abstract fuzzy deduction system. In other words we want to use the following procedure to obtain $u \star (\varphi/a)$:

1. Form the family of *maximal fuzzy subsets of u which are consistent with* (φ/a). We denote this family by $u \perp (\varphi/a)$.[7]

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[5] See also [12].

[6] Hansson [15] considered also an alternative approach, whereby the contraction and expansion steps are carried out in reverse order, i.e., first the base is expanded by φ, and then the resulting base is contracted to be consistent with φ.

[7] In the (crisp) belief revision literature the talk is usually (and equivalently) of the set

2. Select a subset of these by means of a selection function γ: $\gamma(u \perp (\varphi/a)) \subseteq u \perp (\varphi/a)$.
3. Form the meet of the elements of this subset: $\bigsqcap \gamma(u \perp (\varphi/a))$.
4. Join (φ/a) to the result: $u \star (\varphi/a) = (\bigsqcap \gamma(u \perp (\varphi/a))) \sqcup (\varphi/a)$.

We now fill in the details of the above sketched procedure.

3.1. Partial meet fuzzy base revision

First we formally define $u \perp (\varphi/a)$:

DEFINITION 3.1. Let $u \in \mathcal{F}(L)$ and $(\varphi/a) \in L \times W$. Then $u \perp (\varphi/a)$ is the set of elements of $\mathcal{F}(L)$ such that $u' \in u \perp (\varphi/a)$ iff
(i) $u' \sqsubseteq u$,
(ii) $u' \sqcup (\varphi/a)$ is consistent,
(iii) for all $u'' \sqsubseteq u$, if $u' \sqsubset u''$ then $u'' \sqcup (\varphi/a)$ is inconsistent.

Note in particular that if $u \sqcup (\varphi/a)$ is consistent then $u \perp (\varphi/a) = \{u\}$, while if (φ/a) is inconsistent then $u \perp (\varphi/a) = \emptyset$. We need to know that if (φ/a) is consistent then $u \perp (\varphi/a)$ is non-empty. In fact this is the main place where the property of logical compactness of D is required. Under the additional assumption of Zorn's Lemma[8], it enables us to show the following:

PROPOSITION 3.2. *Let $v \in \mathcal{F}(L)$. If $v \sqsubseteq u$ and $v \sqcup (\varphi/a)$ is consistent then there exists $w \in u \perp (\varphi/a)$ such that $v \sqsubseteq w$.*

PROOF. First consider the set $X = \{u' \in \mathcal{F}(L) \mid v \sqsubseteq u' \sqsubseteq u,\ u' \sqcup (\varphi/a) \text{ is consistent}\}$, partially ordered by $\sqsubseteq$. With the help of logical compactness, it can be shown that, for every (non-empty) totally-ordered subset Y of X, the element $\bigsqcup Y$ is an upper-bound for Y *in* X. (If Y is empty then v is an upper-bound for Y in X.) Applying Zorn's Lemma, we then deduce the existence of a maximal element w of X. It can then be shown that for any such w we have both $w \in u \perp (\varphi/a)$ and $v \sqsubseteq w$. ■

Taking $v = u_\top$ in the above proposition gives us the desired non-emptiness for $u \perp (\varphi/a)$:

COROLLARY 3.3. *If (φ/a) is consistent then $u \perp (\varphi/a) \neq \emptyset$.*

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of "maximal subsets which fail to imply $\neg\varphi$", which is denoted by $u \perp \neg\varphi$. We prefer the slightly different notation which does not refer to any connectives.

[8]Every partially ordered set in which every chain (i.e., linearly ordered subset) has an upper bound contains at least one maximal element.

The above proposition is also used in showing the following result, which we need for the proof of Theorem 4.1[9] :

LEMMA 3.4. *Let* $u \in \mathcal{F}(L)$, $\varphi, \varphi' \in L$ *and* $a, a' \in W$. *Then* $u \perp (\varphi/a) = u \perp (\varphi'/a')$ *iff for all* $x \sqsubseteq u$ *we have* $x \sqcup (\varphi/a)$ *is consistent iff* $x \sqcup (\varphi'/a')$ *is consistent.*

PROOF. First we show the "only if" part. Suppose $u \perp (\varphi/a) = u \perp (\varphi'/a')$ and let $x \sqsubseteq u$. Suppose $x \sqcup (\varphi/a)$ is consistent. We will show that also $x \sqcup (\varphi'/a')$ is consistent. But since $x \sqsubseteq u$ and $x \sqcup (\varphi/a)$ is consistent we know, by Proposition 3.2, that there exists $y \in u \perp (\varphi/a)$ such that $x \sqsubseteq y$. Since $u \perp (\varphi/a) = u \perp (\varphi'/a')$ we have $y \in u \perp (\varphi'/a')$ and so, by definition of $u \perp (\varphi'/a')$, we know $y \sqcup (\varphi'/a')$ is consistent. Then, since $x \sqcup (\varphi'/a') \sqsubseteq y \sqcup (\varphi'/a')$, we may apply **Con**$\downarrow$ to deduce $x \sqcup (\varphi'/a')$ is consistent as required. By a symmetrical argument we can show also that if $x \sqcup (\varphi'/a')$ is consistent then $x \sqcup (\varphi/a)$ is consistent.

For the "if" direction suppose that for all $x \sqsubseteq u$ we have $x \sqcup (\varphi/a)$ is consistent iff $x \sqcup (\varphi'/a')$ is consistent. Let $u' \in u \perp (\varphi/a)$. We will show that also $u' \in u \perp (\varphi'/a')$. Since $u' \in u \perp (\varphi/a)$ we know (i) $u' \sqsubseteq u$, (ii) $u' \sqcup (\varphi/a)$ is consistent and (iii) for all $u'' \sqsubseteq u$, if $u' \sqsubset u''$ then $u'' \sqcup (\varphi/a)$ is inconsistent. To show $u' \in u \perp (\varphi'/a')$ we must show (i)′ $u' \sqsubseteq u$, (ii)′ $u' \sqcup (\varphi'/a')$ is consistent and (iii)′ for all $u'' \sqsubseteq u$, if $u' \sqsubset u''$ then $u'' \sqcup (\varphi'/a')$ is inconsistent. Condition (i)′ is just the same as (i), so obviously holds. Then, since $u' \sqsubseteq u$, we know from our assumption that $u' \sqcup (\varphi/a)$ is consistent iff $u' \sqcup (\varphi'/a')$ is consistent. Hence condition (ii)′ follows from condition (ii). To show condition (iii)′ suppose $u'' \sqsubseteq u$ and $u' \sqsubset u''$. Then condition (iii) tells us that $u'' \sqcup (\varphi/a)$ is inconsistent. We may then apply our assumption to deduce that also $u'' \sqcup (\varphi'/a')$ is inconsistent and so (iii)′ holds as required. Hence $u' \in u \perp (\varphi'/a')$ and so we have shown $u \perp (\varphi/a) \subseteq u \perp (\varphi'/a')$. By a symmetrical argument we can also show $u \perp (\varphi'/a') \subseteq u \perp (\varphi/a)$ and so $u \perp (\varphi/a) = u \perp (\varphi'/a')$ as required. ■

We also have the following property (which will be used to prove Proposition 3.8 later). This is one example of a result which is almost trivial in the case of crisp deduction systems, but whose proof in our more general setting requires a little bit more work.

LEMMA 3.5. *Let* $x \in u \perp (\varphi/a)$. *Then* $u \sqcap (\varphi/a) \sqsubseteq x$.

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[9] The proof of this and several other of our results in this paper (including our main result Theorem 4.1) are based on those provided for the special crisp case in [15, 17]. The main difficulty arises from the unavailability in our more general case of the Deduction property.

PROOF. Let $x \in u \perp (\varphi/a)$. Then we know: (i) $x \sqsubseteq u$, (ii) $x \sqcup (\varphi/a)$ is consistent, and (iii) for all $u'' \sqsubseteq u$, $x \sqsubset u''$ implies $u'' \sqcup (\varphi/a)$ is inconsistent. Suppose for contradiction that $u \sqcap (\varphi/a) \not\sqsubseteq x$, equivalently $x \sqcup (u \sqcap (\varphi/a)) \not\sqsubseteq x$. Then, since obviously $x \sqsubseteq x \sqcup (u \sqcap (\varphi/a))$, we have $x \sqsubset x \sqcup (u \sqcap (\varphi/a))$. Since $x \sqsubseteq u$ and $u \sqcap (\varphi/a) \sqsubseteq u$ we also have $x \sqcup (u \sqcap (\varphi/a)) \sqsubseteq u$. Hence we may apply condition (iii) above (substituting $x \sqcup (u \sqcap (\varphi/a))$ for u'') to deduce $x \sqcup (u \sqcap (\varphi/a)) \sqcup (\varphi/a)$ is inconsistent. But $x \sqcup (u \sqcap (\varphi/a)) \sqcup (\varphi/a) = x \sqcup (\varphi/a)$ and so we have that $x \sqcup (\varphi/a)$ is inconsistent. This contradicts condition (ii) above, hence it must be the case that $u \sqcap (\varphi/a) \sqsubseteq x$ as required. ■

We now define selection functions.

DEFINITION 3.6. Let $u \in \mathcal{F}(L)$. A *selection function* for u is a function γ such that for all $(\varphi/a) \in L \times W$, *(i)* if $u \perp (\varphi/a) \neq \emptyset$ then $\emptyset \neq \gamma(u \perp (\varphi/a)) \subseteq u \perp (\varphi/a)$, and *(ii)* if $u \perp (\varphi/a) = \emptyset$ then $\gamma(u \perp (\varphi/a)) = \{u\}$.

Intuitively, selection functions reflect the *resistance to change* of the items of information in u. Given $u \in \mathcal{F}(L)$ and a selection function γ for u we then define a *revision operator* $\star_\gamma$ *for* u as follows:

$$u \star_\gamma (\varphi/a) = \Big(\bigsqcap \gamma(u \perp (\varphi/a))\Big) \sqcup (\varphi/a).$$

DEFINITION 3.7. Let $u \in \mathcal{F}(L)$. Then $\star : L \times W \to \mathcal{F}(L)$ is an operator of *partial meet fuzzy base revision* (for u) iff $\star = \star_\gamma$ for some selection function γ for u.

The following proposition is reminiscent of the Harper Identity from crisp revision [7]. It is used later to prove the soundness of postulate **(F5)** in Theorem 4.1.

PROPOSITION 3.8. *Let γ be a selection function for u. Then* $u \sqcap (u \star_\gamma (\varphi/a)) = \bigsqcap \gamma(u \perp (\varphi/a))$.

PROOF. First we show $\bigsqcap \gamma(u \perp (\varphi/a)) \sqsubseteq u \sqcap (u \star_\gamma (\varphi/a))$. For this we need to show both $\bigsqcap \gamma(u \perp (\varphi/a)) \sqsubseteq u$ and $\bigsqcap \gamma(u \perp (\varphi/a)) \sqsubseteq (u \star_\gamma (\varphi/a))$, i.e., $\bigsqcap \gamma(u \perp (\varphi/a)) \sqsubseteq (\bigsqcap \gamma(u \perp (\varphi/a))) \sqcup (\varphi/a))$. The second statement clearly holds. The first holds since we always have $\gamma(u \perp (\varphi/a)) \neq \emptyset$ (by definition of selection function), and so there exists some $u' \in \gamma(u \perp (\varphi/a))$ such that $u' \sqsubseteq u$ (since $u' \in u \perp (\varphi/a)$). Hence $\bigsqcap \gamma(u \perp (\varphi/a)) \sqsubseteq u' \sqsubseteq u$ as required. To show $u \sqcap (u \star_\gamma (\varphi/a)) \sqsubseteq \bigsqcap \gamma(u \perp (\varphi/a))$ first note that $u \sqcap (u \star_\gamma (\varphi/a)) = u \sqcap (\bigsqcap \gamma(u \perp (\varphi/a)) \sqcup (\varphi/a)) = (u \sqcap \bigsqcap \gamma(u \perp (\varphi/a))) \sqcup (u \sqcap (\varphi/a))$ (using distributivity of the lattice $\mathcal{F}(L)$). Hence to prove our result we need to show that both $(u \sqcap \bigsqcap \gamma(u \perp (\varphi/a))) \sqsubseteq \bigsqcap \gamma(u \perp (\varphi/a))$ and $(u \sqcap (\varphi/a)) \sqsubseteq \bigsqcap \gamma(u \perp (\varphi/a))$. The first of these clearly holds. For the second, note by Lemma 3.5 that $(u \sqcap (\varphi/a)) \sqsubseteq x$ for all $x \in u \perp (\varphi/a)$. Hence $(u \sqcap (\varphi/a)) \sqsubseteq x$ for all $x \in \gamma(u \perp (\varphi/a))$ which gives $u \sqcap (\varphi/a) \sqsubseteq \bigsqcap \gamma(u \perp (\varphi/a))$ as required. ∎

The above result says that $u \sqcap (u \star_\gamma (\varphi/a))$ may be equated with the result of "contracting" u to make room for the new item (φ/a). We will return briefly to this point in Section 6.

3.2. Examples

Let us give an example of partial meet fuzzy base revision "in action" for each of the instantiations of the framework we gave in Section 2.1.

3.2.1. Crisp deduction systems

For crisp deduction systems the operation reduces to the usual partial meet base revision from [15]. For example suppose $u = \{x, x \to y, z\}$ and suppose we receive the new information $\neg(y \wedge z)$ (it is understood that all the stated formulas have degree 1). Then we get

$$u \perp (\neg(y \wedge z)) = \{\{x, x \to y\}, \{x, z\}, \{x \to y, z\}\}.$$

Suppose our selection function γ selects the first two subsets above:

$$\gamma(u \perp (\neg(y \wedge z)) = \{\{x, x \to y\}, \{x, z\}\}.$$

Then we get

$$u \star_\gamma \neg(y \wedge z) = \left(\bigcap \gamma(u \perp (\neg(y \wedge z))\right) \cup \{\neg(y \wedge z)\} = \{x, \neg(y \wedge z)\}.$$

3.2.2. Łukasiewicz fuzzy logic

Suppose u_0 is given as in Section 2.1.2, i.e.,

$$u_0 = \{(x/0.75), (x \to y/0.75), (z/0.25)\}.$$

Then suppose we receive the new information $(\neg(y \wedge z)/1)$, i.e., it is definitely *not* the case that y and z are true together. We know from the remark at the end of Section 2.1.2 that u_0 is inconsistent with this new information. In order to make u_0 consistent with $(\neg(y \wedge z)/1)$ we need to modify it so that $D_{\text{luk}}(u_0)(y \wedge z) = 0$. This can be achieved either by holding the truth-degrees of x and $x \to y$ fixed while lowering that of z to 0, or by holding the truth-degree of z fixed and lowering that of either x or $x \to y$ (or both) just enough to ensure $D_{\text{luk}}(u_0)(y) = 0$. Precisely, we can show that

$$\begin{aligned} u_0 \perp (\neg(y \wedge z)/1) = & \{\{(x/0.75), (x \to y/0.75)\}\} \cup \\ & \{u' \sqsubseteq u_0 \mid \ 0.25 \leq u'(x), \\ & \qquad u'(x \to y) = 1 - u'(x), \\ & \qquad u'(z) = u_0(z)\}. \end{aligned}$$

Suppose we prefer to keep the information item $(x/0.75)$, and that this is reflected by applying the selection function

$$\begin{aligned} \gamma(u_0 \perp (\neg(y \wedge z)/1)) &= \{u' \in u_0 \perp (\neg(y \wedge z)/1) \mid u'(x) = u_0(x)\} \\ &= \big\{\{(x/0.75), (x \to y/0.75)\}, \\ &\quad \{(x/0.75), (x \to y/0.25), (z/0.25)\}\big\}. \end{aligned}$$

Then, using u^* as shorthand for $u_0 \star_\gamma (\neg(y \wedge z)/1)$, we have $u^*(\neg(y \wedge z)) = 1$, while for $\theta \neq \neg(y \wedge z)$ we have

$$\begin{aligned} u^*(\theta) &= \Big[\prod \gamma(u_0 \perp (\neg(y \wedge z)/1))\Big](\theta) \\ &= \inf\{u'(\theta) \mid u' \in \gamma(u_0 \perp (\neg(y \wedge z)/1))\}. \end{aligned}$$

Hence, as our final result we get:

$$u_0 \star_\gamma (\neg(y \wedge z)/1) = \{(x/0.75), (x \to y/0.25), (\neg(y \wedge z)/1)\}.$$

3.2.3. Necessity logic

Let u_0 be as in the previous example and suppose we get the new information $(\neg(y \wedge z)/0.25)$. Then, since $Supp(u_0 \sqcup (\neg(y \wedge z)/0.25)) = \{x, x \to y, z, \neg(y \wedge z)\}$ is Cn-inconsistent we know $u_0 \sqcup (\neg(y \wedge z)/0.25)$ is inconsistent. Finding the fuzzy subsets of u_0 which are maximally consistent with $(\neg(y \wedge z)/0.25)$ essentially reduces to finding the crisp subsets of $Supp(u_0)$ which are maximally Cn-consistent with $\neg(y \wedge z)$:

$$\begin{aligned} u_0 \perp (\neg(y \wedge z)/0.25) = \ & \big\{\{(x \to y/0.75), (z/0.25)\}, \\ & \{(x/0.75), (z/0.25)\}, \\ & \{(x/0.75), (x \to y/0.75)\}\big\}. \end{aligned}$$

Hence so far this doesn't look much different from the case of crisp deduction systems. The only difference is that now not all the formulas have degree

1. We have the option of using this extra expressiveness to actually help *define* a selection function, perhaps according to a principle that formulas with greater degrees should be kept whenever possible. Indeed this is the approach usually taken in works on belief revision within possibility theory such as [4, 5]. For instance in the above example we could prefer to throw out the information item with the lowest degree, i.e., $(z/0.25)$. This would be reflected by using a selection function for u_0 such that:

$$\gamma(u_0 \perp (\neg(y \wedge z)/0.25)) = \{\{(x/0.75), (x \to y/0.75)\}\}.$$

Then

$$\bigsqcap(\gamma(u_0 \perp (\neg(y \wedge z)/0.25))) = \gamma(u_0 \perp (\neg(y \wedge z)/0.25))$$

and so $u_0 \star_\gamma (\neg(y \wedge z)/0.25) = \{(x/0.75), (x \to y/0.75), (\neg(y \wedge z)/0.25)\}$.[10] We remark, however, that there is nothing to stop us from defining γ independently of the degrees.[11]

3.2.4. Probability logic (lower envelopes)

For a probabilistic example let us again use the base u_0 from earlier and suppose this time we get new information $(\neg y/0.75)$ which, since as we saw in Section 2.1.4, $D_{\mathrm{P}}(u_0)(y) > 0.25$, is inconsistent with u_0. Then it can be shown that

$$\begin{aligned} u_0 \perp (\neg y/0.75) = \{u' \sqsubseteq u_0 \mid\ & 0.5 \leq u'(x), \\ & u'(x \to y) = 1.25 - u'(x), \\ & u'(z) = u_0(z)\}. \end{aligned}$$

Suppose our selection function γ is defined by

$$\gamma(u \perp (\neg y/0.75)) = \{u' \in u \perp (\neg y/0.75) \mid 0.6 \leq u'(x)\},$$

reflecting a certain "level of security" behind the particular item of information $(x/0.75)$: we are not willing to choose any subset of u_0 in which the probability of x falls below 0.6. Then, using u^* now as shorthand for $u_0 \star_\gamma (\neg y/0.75)$ we have $u^*(\neg y) = 0.75$, while for $\theta \neq \neg y$ we have

$$\begin{aligned} u^*(\theta) &= \left[\bigsqcap \gamma(u_0 \perp (\neg y/0.75))\right](\theta) \\ &= \inf\{u'(\theta) \mid u' \in \gamma(u_0 \perp (\neg y/0.75))\}. \end{aligned}$$

Hence $u_0 \star_\gamma (\neg y/0.75) = \{(x/0.6),\ (x \to y/0.5),\ (z/0.25), (\neg y/0.75)\}$.

[10] For a related approach see [23].

[11] In fact the question of the precise nature of the relationship between degrees of confidence (i.e., degrees for us) and degree of resistance to change is one of the open philosophical problems in belief revision recently posed by Hansson [16].

4. Characterising partial meet fuzzy base revision

In this section we axiomatically characterise the class of partial meet fuzzy base revision operators. It turns out that the class is characterised by the following five postulates, each of which generalises a postulate from the corresponding axiomatisation from the crisp case [15]. We list the usual names of these corresponding postulates to the right.

(F1) $a \leq_W [u \star (\varphi/a)](\varphi)$ (Success)

(F2) $u \star (\varphi/a)$ is consistent if (φ/a) is consistent (Consistency)

(F3) $u \star (\varphi/a) \sqsubseteq u \sqcup (\varphi/a)$ (Inclusion)

(F4) For all $\theta \in L$, $b \in W$, if $b \not\leq_W [u \star (\varphi/a)](\theta)$ and $b \leq_W u(\theta)$ then there exists u' such that $u \star (\varphi/a) \sqsubseteq u' \sqsubseteq u \sqcup (\varphi/a)$, u' is consistent and $u' \sqcup (\theta/b)$ is inconsistent. (Relevance)

(F5) If, for all $x \sqsubseteq u$, we have $x \sqcup (\varphi/a)$ is consistent iff $x \sqcup (\varphi'/a')$ is consistent, then $u \sqcap (u \star (\varphi/a) =$ $u \sqcap (u \star (\varphi'/a'))$. (Uniformity)

(F1) says that the revision is *successful*, i.e., that after revision by (φ/a), the formula φ is assigned a degree of at least a. **(F2)** requires the result of revision to be consistent, provided the input is itself consistent. **(F3)** says that the revised base should not contain more information than that obtained by simply joining the original base with the new information. **(F4)** seeks to minimise unnecessary loss of information. Roughly, it expresses that if, for every consistent fuzzy base u' lying between $u \star (\varphi/a)$ and $u \sqcup (\varphi/a)$, it is possible to raise the degree of θ from $u'(\theta)$ to b without incurring inconsistency, then there is no reason for the revised degree of θ to fall below b. An interesting special case of this rule can be found by substituting $u(\theta)$ for b:

(F4′) For all $\theta \in L$, if $u(\theta) \not\leq_W [u \star (\varphi/a)](\theta)$ then there exists u' such that $u \star (\varphi/a) \sqsubseteq u' \sqsubseteq u \sqcup (\varphi/a)$, u' is consistent and $u' \sqcup (\theta/u(\theta))$ is inconsistent.

It is easy to see that, in the crisp case, **(F4′)** and **(F4)** coincide. **(F4′)** is a way of saying that the degree of θ is reduced only if keeping it at $u(\theta)$ somehow contributes to the inconsistency of u with (φ/a). Finally for **(F5)**, first note that $u \sqcap (u \star (\varphi/a))$ can be understood as that information in u which is retained in $u \star (\varphi/a)$. Hence **(F5)** says that if two different inputs are consistent with precisely the same fuzzy subsets of u then they remove the same information from u. We now give the central result of the paper, which generalises the characterisation given in [15] for crisp deduction systems.

Theorem 4.1. *Let $u \in \mathcal{F}(L)$ and $\star$ be an operator for u. Then $\star$ is an operator of partial meet fuzzy base revision for u iff $\star$ satisfies* **(F1)**–**(F5)**.

Proof (Soundness). Let γ be a selection function for u. We check that $\star_\gamma$ satisfies each postulate in turn:

(F1) We have $(\varphi/a) \sqsubseteq \left(\bigsqcap \gamma(u \perp (\varphi/a))\right) \sqcup (\varphi/a) = u \star_\gamma (\varphi/a)$. Hence $a \leq_W [u \star_\gamma (\varphi/a)](\varphi)$ as required.
(F2) Suppose (φ/a) is consistent and choose $u' \in \gamma(u \perp (\varphi/a))$. (Such a u' exists since $\gamma(u \perp (\varphi/a))$ is always non-empty by definition.) Then, since $\bigsqcap \gamma(u \perp (\varphi/a)) \sqsubseteq u'$, we get $\left(\bigsqcap \gamma(u \perp (\varphi/a))\right) \sqcup (\varphi/a) \sqsubseteq u' \sqcup (\varphi/a)$. Since (φ/a) is consistent we know $u \perp (\varphi/a) \neq \emptyset$ (by Corollary 3.3) and so $\gamma(u \perp (\varphi/a)) \subseteq u \perp (\varphi/a)$ by definition of selection function. Hence $u' \in u \perp (\varphi/a)$, which means that $u' \sqcup (\varphi/a)$ is consistent. Hence, by **Con**$\downarrow$, so is

$$\left(\bigsqcap \gamma(u \perp (\varphi/a))\right) \sqcup (\varphi/a) = u \star_\gamma (\varphi/a).$$

Thus **(F2)** is satisfied.
(F3) If (φ/a) is inconsistent then $u \perp (\varphi/a) = \emptyset$ and so $\gamma(u \perp (\varphi/a)) = \{u\}$ by definition of selection function. Hence in this case $u \star_\gamma (\varphi/a) = u \sqcup (\varphi/a)$ and so **(F3)** certainly holds. So suppose (φ/a) is consistent. Then, as in the proof of **(F2)** above, we may choose $u' \in \gamma(u \perp (\varphi/a))$ such that $u \star_\gamma (\varphi/a) = \left(\bigsqcap \gamma(u \perp (\varphi/a))\right) \sqcup (\varphi/a) \sqsubseteq u' \sqcup (\varphi/a)$. Since $u' \in u \perp (\varphi/a)$ we know that $u' \sqsubseteq u$. Hence $u \star_\gamma (\varphi/a) \sqsubseteq u' \sqcup (\varphi/a) \sqsubseteq u \sqcup (\varphi/a)$, which shows that **(F3)** is satisfied.
(F4) Let $\theta \in L$ and $b \in W$ be such that $b \not\leq_W [u \star_\gamma (\varphi/a)](\theta)$ and $b \leq_W u(\theta)$. We must find some u' such that (i) $u \star_\gamma (\varphi/a) \sqsubseteq u' \sqsubseteq (u \sqcup (\varphi/a))$, (ii) u' is consistent, and (iii) $(u' \sqcup (\theta/b))$ is inconsistent. But from $b \not\leq_W [u \star_\gamma (\varphi/a)](\theta)$ we infer $b \not\leq_W [\bigsqcap \gamma(u \perp (\varphi/a))](\theta)$ (since $\bigsqcap \gamma(u \perp (\varphi/a)) \sqsubseteq \left(\bigsqcap \gamma(u \perp (\varphi/a))\right) \sqcup (\varphi/a) = u \star_\gamma (\varphi/a)$). This is equivalent to saying

$$b \not\leq_W \inf\{x(\theta) \mid x \in \gamma(u \perp (\varphi/a))\},$$

hence there must exist $x' \in \gamma(u \perp (\varphi/a))$ such that $b \not\leq_W x'(\theta)$. We now claim that the fuzzy base $x' \sqcup (\varphi/a)$ satisfies the required conditions (i)–(iii) above. To see that (i) is satisfied we have $\bigsqcap \gamma(u \perp (\varphi/a)) \sqsubseteq x'$ (since $x' \in \gamma(u \perp (\varphi/a))$), while also $x' \sqsubseteq u$ (by definition of $u \perp (\varphi/a)$). Hence we get $u \star_\gamma (\varphi/a) = \left(\bigsqcap \gamma(u \perp (\varphi/a))\right) \sqcup (\varphi/a) \sqsubseteq (x' \sqcup (\varphi/a)) \sqsubseteq (u \sqcup (\varphi/a))$ as required. For (ii) we know $(x' \sqcup (\varphi/a))$ is consistent by definition of

$u \perp (\varphi/a)$. Finally for (iii) we know, again by definition of $u \perp (\varphi/a)$, that for all $u'' \sqsubseteq u$, if $x' \sqsubset u''$ then $u'' \sqcup (\varphi/a)$ is inconsistent. Hence to show $x' \sqcup (\varphi/a) \sqcup (\theta/b)$ is inconsistent it is sufficient to show that both $(x' \sqcup (\theta/b)) \sqsubseteq u$ and $x' \sqsubset (x' \sqcup (\theta/b))$. That this first condition holds follows since we know already that $x' \sqsubseteq u$, while $(\theta/b) \sqsubseteq u$ follows from the original assumption that $b \leq_W u(\theta)$. For the second condition we clearly have $x' \sqsubseteq (x' \sqcup (\theta/b))$, so it remains to show that $(x' \sqcup (\theta/b)) \not\sqsubseteq x'$, equivalently, $(\theta/b) \not\sqsubseteq x'$. But we know $b \not\leq_W x'(\theta)$, i.e., $[\theta/b](\theta) \not\leq_W x'(\theta)$, hence $(\theta/b) \not\sqsubseteq x'$ as required. This completes the proof of **(F4)**.

(F5) Suppose that, for all $x \sqsubseteq u$, we have that $x \sqcup (\varphi/a)$ is consistent iff $x \sqcup (\varphi'/a')$ is consistent. Then, by Lemma 3.4, this is equivalent to saying $u \perp (\varphi/a) = u \perp (\varphi'/a')$. We then have $u \sqcap (u \star_\gamma (\varphi/a)) = \bigsqcap \gamma(u \perp (\varphi/a))$ (by Proposition 3.8) $= \bigsqcap \gamma(u \perp (\varphi'/a')) = u \sqcap (u \star_\gamma (\varphi'/a'))$ as required. ■

Proof (Completeness). Let $\star$ be an operator for u which satisfies **(F1)**–**(F5)**. We must find some selection function γ for u such that $\star = \star_\gamma$. We define γ from u and $\star$ by setting, for each $(\varphi/a) \in L \times W$ [12],

$$\gamma(u \perp (\varphi/a)) = \begin{cases} \{u' \in u \perp (\varphi/a) \mid u \sqcap (u \star (\varphi/a)) \sqsubseteq u'\} & \text{if } u \perp (\varphi/a) \neq \emptyset \\ \{u\} & \text{otherwise.} \end{cases}$$

There are now three things we must show: 1. γ is well-defined, 2. γ is a selection function for u, i.e., γ fulfills the conditions of Definition 3.6, and 3. $u \star_\gamma (\varphi/a) = u \star (\varphi/a)$.

1. First we need to make sure γ is well-defined, i.e., that if $u \perp (\varphi/a) = u \perp (\varphi'/a')$ then applying γ to $u \perp (\varphi/a)$ returns the same result as applying γ to $u \perp (\varphi'/a')$. So suppose $u \perp (\varphi/a) = u \perp (\varphi'/a')$. Clearly if both $u \perp (\varphi/a)$ and $u \perp (\varphi'/a')$ are empty then $\gamma(u \perp (\varphi/a)) = \gamma(u \perp (\varphi'/a')) = \{u\}$ as required. So suppose both are non-empty. We need to show that, for each $u' \in u \perp (\varphi/a)$, we have $u \sqcap (u \star (\varphi/a)) \sqsubseteq u'$ iff $u \sqcap (u \star (\varphi'/a')) \sqsubseteq u'$. But, by Lemma 3.4, $u \perp (\varphi/a) = u \perp (\varphi'/a')$ gives us that, for all $x \sqsubseteq u$, $x \sqcup (\varphi/a)$ is consistent iff $x \sqcup (\varphi'/a')$ is consistent. Using this with the fact that $\star$ satisfies **(F5)** then gives us $u \sqcap (u \star (\varphi/a)) = u \sqcap (u \star (\varphi'/a'))$ which suffices.

2. Next we show γ is a selection function. Condition *(ii)* from Definition 3.6 is obviously satisfied. To verify condition *(i)* of the definition, let's suppose $u \perp (\varphi/a) \neq \emptyset$. We must show $\emptyset \neq \gamma(u \perp (\varphi/a)) \subseteq u \perp (\varphi/a)$. Clearly we have $\gamma(u \perp (\varphi/a)) \subseteq u \perp (\varphi/a)$, so it remains to show $\gamma(u \perp (\varphi/a)) \neq \emptyset$. By Proposition 3.2, for this it suffices to show $(u \sqcap (u \star (\varphi/a))) \sqcup (\varphi/a)$ is

[12] The construction here, although inspired by [15, 17], actually mimics the one used in the original AGM paper [1].

consistent. We have

$$(u \sqcap (u \star (\varphi/a))) \sqcup (\varphi/a) \sqsubseteq (u \star (\varphi/a)) \sqcup (\varphi/a).$$

But, since $a \leq_W [u \star (\varphi/a)](\varphi)$ (by **(F1)**), equivalently $(\varphi/a) \sqsubseteq u \star (\varphi/a)$, we may deduce that $(u \star (\varphi/a)) \sqcup (\varphi/a) = u \star (\varphi/a)$ and so $(u \sqcap (u \star (\varphi/a))) \sqcup (\varphi/a) \sqsubseteq u \star (\varphi/a)$. Now since $u \perp (\varphi/a) \neq \emptyset$ it must be the case that (φ/a) is consistent. Hence $u \star (\varphi/a)$ is consistent (by **(F2)**) and so we may deduce from **Con**$\downarrow$ that also $(u \sqcap (u \star (\varphi/a))) \sqcup (\varphi/a)$ is consistent as required.

3. We consider two cases:

Case *(i)*: $u \perp (\varphi/a) = \emptyset$. In this case we have $\gamma(u \perp (\varphi/a)) = \{u\}$ and so $u \star_\gamma (\varphi/a) = u \sqcup (\varphi/a)$. Meanwhile, since $u \perp (\varphi/a) = \emptyset$ it must be the case that (φ/a) is inconsistent by Corollary 3.3. Hence, since $\star$ satisfies the property **(F9)** derivable from **(F1)**–**(F5)**(see Proposition 4.2) we get also $u \star (\varphi/a) = u \sqcup (\varphi/a)$ as required.

Case *(ii)*: $u \perp (\varphi/a) \neq \emptyset$. First we show $u \star (\varphi/a) \sqsubseteq u \star_\gamma (\varphi/a)$. By **(F3)** we have $u \star (\varphi/a) \sqsubseteq u \sqcup (\varphi/a)$ and so we get $u \star (\varphi/a) = (u \sqcup (\varphi/a)) \sqcap (u \star (\varphi/a)) =$ (by **(F1)**) $(u \sqcup (\varphi/a)) \sqcap (u \star (\varphi/a) \sqcup (\varphi/a)) = (u \sqcap (u \star (\varphi/a))) \sqcup (\varphi/a)$ (this last step uses distributivity of the lattice $\mathcal{F}(L)$). Hence it suffices to show $(u \sqcap (u \star (\varphi/a))) \sqcup (\varphi/a) \sqsubseteq u \star_\gamma (\varphi/a)$, i.e.,

$$(u \sqcap (u \star (\varphi/a))) \sqcup (\varphi/a) \sqsubseteq \left(\bigsqcap \gamma(u \perp (\varphi/a))\right) \sqcup (\varphi/a).$$

To show this it is enough to show $u \sqcap (u \star (\varphi/a)) \sqsubseteq \bigsqcap \gamma(u \perp (\varphi/a))$. But this is true since, by definition of γ, we have that $u \sqcap (u \star (\varphi/a)) \sqsubseteq u'$ for all $u' \in \gamma(u \perp (\varphi/a))$. Thus $u \star (\varphi/a) \sqsubseteq u \star_\gamma (\varphi/a)$ as required.

Now we show $u \star_\gamma (\varphi/a) \sqsubseteq u \star (\varphi/a)$, i.e.,

$$\left(\bigsqcap \gamma(u \perp (\varphi/a))\right) \sqcup (\varphi/a) \sqsubseteq u \star (\varphi/a).$$

By **(F1)** we have $(\varphi/a) \sqsubseteq u \star (\varphi/a)$, thus it remains to show that also $(\bigsqcap \gamma(u \perp (\varphi/a))) \sqsubseteq u \star (\varphi/a)$, i.e., that, for all $\theta \in L$, we have

$$\left[\bigsqcap \gamma(u \perp (\varphi/a))\right](\theta) \leq_W [u \star (\varphi/a)](\theta).$$

We prove this by contradiction. So suppose there existed some $\theta \in L$ such that

$$\left[\bigsqcap \gamma(u \perp (\varphi/a))\right](\theta) \not\leq_W [u \star (\varphi/a)](\theta).$$

Let us write $[\bigsqcap \gamma](\theta)$ as an abbreviation for the left-hand side of this inequality. Clearly, since $(\bigsqcap \gamma(u \perp (\varphi/a)) \sqsubseteq u$, we also have $[\bigsqcap \gamma](\theta) \leq_W u(\theta)$.

Thus we may apply the postulate **(F4)** (substituting $[\bigsqcap\gamma](\theta)$ for b there) to deduce the existence of some u' such that (i) $u \star (\varphi/a) \sqsubseteq u' \sqsubseteq (u \sqcup (\varphi/a))$, (ii) u' is consistent and (iii) $u' \sqcup (\theta/[\bigsqcap\gamma](\theta))$ is inconsistent.

Now since $u' \sqsubseteq (u \sqcup (\varphi/a))$ (and obviously $u' \sqsubseteq (u' \sqcup (\varphi/a))$) we have $u' \sqsubseteq (u' \sqcup (\varphi/a)) \sqcap (u \sqcup (\varphi/a)) = (u' \sqcap u) \sqcup (\varphi/a)$ (using distributivity of $\mathcal{F}(L)$). Hence, since $u' \sqcup (\theta/[\bigsqcap\gamma](\theta))$ is inconsistent, $(u' \sqcap u) \sqcup (\theta/[\bigsqcap\gamma](\theta)) \sqcup (\varphi/a)$ is inconsistent as well (using **Con**$\downarrow$). From this we can conclude that there can be no $y \in \gamma(u \perp (\varphi/a))$ such that $(u' \sqcap u) \sqsubseteq y$. For if there was such a y then we would have $(u' \sqcap u) \sqcup (\theta/[\bigsqcap\gamma](\theta)) \sqsubseteq y$ and so $y \sqcup (\varphi/a)$ would be inconsistent by **Con**$\downarrow$, contradicting $y \in u \perp (\varphi/a)$. Nonetheless we show that our current assumptions allow us also to infer the existence of such a y, giving the required contradiction to complete the proof. To see this, first note that we have both $u' \sqcap u \sqsubseteq u$ and $(u' \sqcap u) \sqcup (\varphi/a)$ is consistent. This second part holds since clearly $(u' \sqcap u) \sqsubseteq u'$, while also $(\varphi/a) \sqsubseteq u'$ (since $a \leq_W [u \star (\varphi/a)](\varphi)$ by **(F1)** and $u \star (\varphi/a) \sqsubseteq u'$, thus $a \leq_W u'(\varphi)$). Thus $(u' \sqcap u) \sqcup (\varphi/a) \sqsubseteq u'$ and we conclude using **Con**$\downarrow$ from the fact that u' is consistent. Hence, by Proposition 3.2, there exists $y \in u \perp (\varphi/a)$ such that $(u' \sqcap u) \sqsubseteq y$. But, using the fact that $u \star (\varphi/a) \sqsubseteq u'$, we get that $(u \sqcap (u \star (\varphi/a))) \sqsubseteq (u' \sqcap u)$. Hence we may deduce that also $(u \sqcap (u \star (\varphi/a))) \sqsubseteq y$ and so, by definition of γ, $y \in \gamma(u \perp (\varphi/a)$ as required. ■

The next proposition gives us some more rules which can be derived from **(F1)**–**(F5)** (in fact with **(F4′)** replacing **(F4)**) and thus are properties of partial meet fuzzy base revision.

PROPOSITION 4.2. *Let $u \in \mathcal{F}(L)$ and $\star$ be an operator for u which satisfies* **(F1)**–**(F3)**, **(F4′)** *and* **(F5)**. *Then $\star$ also satisfies the following properties:*

(F6) *If $u \sqcup (\varphi/a)$ is consistent then $u \star (\varphi/a) = u \sqcup (\varphi/a)$.*

(F7) *If u is consistent and $a \leq_W u(\varphi)$ then $u \star (\varphi/a) = u$.*

(F8) *If $u(\varphi) \leq_W a$ then $[u \star (\varphi/a)](\varphi) = a$.*

(F9) *If (φ/a) is inconsistent then $u \star (\varphi/a) = u \sqcup (\varphi/a)$.*

(F10) *If $D(\varphi/a) = D(\varphi'/a')$ then $u \sqcap (u \star (\varphi/a)) = u \sqcap (u \star (\varphi'/a'))$.*

PROOF. **(F6)** Suppose $u \sqcup (\varphi/a)$ is consistent. We know already from **(F3)** that $u \star (\varphi/a) \sqsubseteq u \sqcup (\varphi/a)$. Hence it remains to show that also $u \sqcup (\varphi/a) \sqsubseteq u \star (\varphi/a)$. But suppose for contradiction that $u \sqcup (\varphi/a) \not\sqsubseteq u \star (\varphi/a)$. Then, since $(\varphi/a) \sqsubseteq u \star (\varphi/a)$ by **(F1)**, this gives $u \not\sqsubseteq u \star (\varphi/a)$, and so there exists $\theta \in L$ such that $u(\theta) \not\leq_W [u \star (\varphi/a)](\theta)$. By **(F4′)** there exists some u' such that (among other things) $u' \sqsubseteq u \sqcup (\varphi/a)$ and $u' \sqcup (\theta/u(\theta))$ is inconsistent. Since clearly also $(\theta/u(\theta)) \sqsubseteq u \sqcup (\varphi/a)$ we have $u' \sqcup (\theta/u(\theta)) \sqsubseteq u \sqcup (\varphi/a)$ and so, by **Con**$\downarrow$, we deduce $u \sqcup (\varphi/a)$ is inconsistent. This gives the required contradiction, hence $u \sqcup (\varphi/a) \sqsubseteq u \star (\varphi/a)$ as required.

(F7) This follows from **(F6)** and by noting that $a \leq_W u(\varphi)$ implies $u = u \sqcup (\varphi/a)$.

(F8) If $u(\varphi) \leq_W a$ then $[u \sqcup (\varphi/a)](\varphi) = a$. **(F1)** gives $a \leq_W [u \star (\varphi/a)](\varphi)$, **(F3)** gives $[u \star (\varphi/a)](\varphi) \leq_W [u \sqcup (\varphi/a)](\varphi) = a$ and thus equality.

(F9) Suppose (φ/a) is inconsistent. Then by **(F1)** and **Con**$\downarrow$ we have $u \star (\varphi/a)$ is inconsistent and hence (again using **Con**$\downarrow$) there can be no consistent u' with $u \star (\varphi/a) \sqsubseteq u' \sqsubseteq u \sqcup (\varphi/a)$. By **(F4′)** it follows that we must have $u \sqsubseteq u \star (\varphi/a)$. Hence, since also $(\varphi/a) \sqsubseteq u \star (\varphi/a)$ by **(F1)**, we get $u \sqcup (\varphi/a) \sqsubseteq u \star (\varphi/a)$. We obtain equality by **(F3)**.

(F10) If $D(\varphi/a) = D(\varphi'/a')$ then it holds for all $v \sqsubseteq u$ that $D(v \sqcup (\varphi/a)) = D(v \sqcup D(\varphi/a)) = D(v \sqcup D(\varphi'/a')) = D(v \sqcup (\varphi'/a'))$. (Note we apply here the property that $D(w_1 \sqcup w_2) = D(w_1 \sqcup D(w_2))$ for all fuzzy bases w_1, w_2 which follows from three generalised Tarski properties.) Thus $v \sqcup (\varphi/a)$ is consistent iff $v \sqcup (\varphi'/a')$ is consistent. Application of **(F5)** then yields $u \sqcap (u \star (\varphi/a)) = u \sqcap (u \star (\varphi'/a'))$ as required. ■

The above derived properties can be explained as follows. **(F6)** is a "vacuity" property which says that if the new information (φ/a) is consistent with the current information u, then the new base is formed by simply adding (φ/a) to u. As a consequence of this we get **(F7)**, which says that if u is consistent and φ is already explicitly assigned a degree in u of at least a then revising by (φ/a) leaves the base unchanged. **(F8)** says that if $u(\varphi) \leq_W a$ then φ is assigned a degree in the new base of *precisely* a. **(F9)** states that if the new information is inconsistent then the new base is again formed by just adding it to the current information. Finally, **(F10)** says that revising by information which is "logically equivalent" removes the same information from u. Note that, for the common case when W is linearly ordered, **(F7)** and **(F8)** together give:

$$\text{If } u \text{ is consistent then } [u \star (\varphi/a)](\varphi) = u(\varphi) \curlyvee a.$$

However, partial meet fuzzy base revision operators do *not* satisfy this property in general, as the following example shows.

EXAMPLE 4.3. We consider the abstract fuzzy deduction system (L, W, D), where *(i)* we assume for simplicity that L contains just a single element $\hat{\varphi}$, *(ii)* we let $W = \mathcal{P}(\{0, 1\})$, ordered by set inclusion, and *(iii)* we define D to be just the identity function. So note a fuzzy base here will just take the form $(\hat{\varphi}/A)$ for some (possibly empty) $A \subseteq \{0,1\}$, with the only inconsistent fuzzy base being $(\hat{\varphi}/\{0,1\})$. Note also that D so-defined trivially satisfies Reflexivity, Monotony and Idempotence. It is also quite straightforward to show D is logically compact, thus (L, W, D) is a legitimate abstract fuzzy deduction system. Now let u be the consistent fuzzy base $(\hat{\varphi}/\{0\})$ and suppose we want to revise u by the new information $(\hat{\varphi}/\{1\})$ using some partial meet revision operator $\star_\gamma$ for u. Note that $u(\hat{\varphi}) \Cup \{1\} = u(\hat{\varphi}) \cup \{1\} = \{0,1\}$. Hence if it were the case that $[u \star_\gamma (\hat{\varphi}/\{1\})](\hat{\varphi}) = u(\hat{\varphi}) \Cup \{1\}$ then we would have $u \star_\gamma (\hat{\varphi}/\{1\})$ is inconsistent, contradicting the postulate **(F2)**. Hence the above-mentioned property is not satisfied here. (In fact since $u \perp (\hat{\varphi}/\{1\})$ here contains only the single element $(\hat{\varphi}/\emptyset)$, the only possible result for $u \star_\gamma (\hat{\varphi}/\{1\})$ is $(\hat{\varphi}/\emptyset) \sqcup (\hat{\varphi}/\{1\}) = (\hat{\varphi}/\{1\})$.)

5. Limiting cases

In this section we consider two limiting cases of partial meet fuzzy base revision, both of which are familiar from the special crisp case [2, 15, 17]. In *full meet* fuzzy base revision the selection function γ selects *all* elements of $u \perp (\varphi/a)$, while in *maxichoice* fuzzy base revision, γ selects just a single element of $u \perp (\varphi/a)$. We will give axiomatic characterisations for each in turn.

5.1. Full meet fuzzy base revision

The operator of full meet fuzzy base revision (for u) $\star_{fm}$ is the special case in which the selection function γ selects *all* elements of $u \perp (\varphi/a)$ if $u \perp (\varphi/a) \neq \emptyset$, while if $u \perp (\varphi/a) = \emptyset$ then $\gamma(u \perp (\varphi/a)) = \{u\}$ as before, i.e.,

$$u \star_{fm} (\varphi/a) = \begin{cases} \left(\bigsqcap(u \perp (\varphi/a))\right) \sqcup (\varphi/a) & \text{if } u \perp (\varphi/a) \neq \emptyset \\ u \sqcup (\varphi/a) & \text{otherwise.} \end{cases}$$

This operator may be characterised with the help of the following rule:

(FM) For all consistent $v \in \mathcal{F}(L)$ such that $(\varphi/a) \sqsubseteq v \sqsubseteq (u \sqcup (\varphi/a))$, we have $v \sqcup (u \star (\varphi/a))$ is consistent.

By taking $v = (\varphi/a)$ above, we can see that, in the presence of **(F1)**, **(FM)** implies the "consistency" postulate **(F2)**. We obtain the following representation theorem for full meet fuzzy base revision:

THEOREM 5.1. *Let $u \in \mathcal{F}(L)$ and let $\star$ be an operator for u. Then $\star = \star_{fm}$ iff $\star$ satisfies* **(F1)**, **(F3)**–**(F5)** *and* **(FM)**.

Proof (*Outline*). Soundness: Since $\star_{fm}$ is a special case of partial meet fuzzy base revision, we already know from Theorem 4.1 that **(F1)**, **(F3)**–**(F5)** are satisfied, so it remains to show **(FM)**. So let v be such that $(\varphi/a) \sqsubseteq v \sqsubseteq (u \sqcup (\varphi/a))$ and v is consistent. We must show $v \sqcup (u \star_{fm} (\varphi/a))$ is consistent. Using these inequalities with the distributivity of $\mathcal{F}(L)$, we may re-express v as $v = (v \sqcap u) \sqcup (\varphi/a)$. Hence $(v \sqcap u) \sqcup (\varphi/a)$ is consistent. Using this we may apply Proposition 3.2 and deduce the existence of some $w \in (u \perp (\varphi/a))$ such that $(v \sqcap u) \sqsubseteq w$. Now since v is consistent and $(\varphi/a) \sqsubseteq v$ we know (φ/a) is consistent by **Con**$\downarrow$ and so $(u \perp (\varphi/a)) \neq \emptyset$ by Corollary 3.3. Hence by definition of $\star_{fm}$ we have $u \star_{fm} (\varphi/a) = (\bigsqcap(u \perp (\varphi/a))) \sqcup (\varphi/a)$. Using this with the above re-expression of v we get $v \sqcup (u \star_{fm} (\varphi/a)) = (v \sqcap u) \sqcup \left(\bigsqcap(u \perp (\varphi/a))\right) \sqcup (\varphi/a)$. From this and the properties of w we see $v \sqcup (u \star_{fm} (\varphi/a)) \sqsubseteq w \sqcup (\varphi/a)$. This latter is consistent so, by **Con**$\downarrow$, so is $v \sqcup (u \star_{fm} (\varphi/a))$ as required.

Completeness: We already know by Theorem 4.1 that if $\star$ satisfies **(F1)**–**(F5)** then $\star$ is a partial meet fuzzy base revision operator and so $\star = \star_\gamma$ for *some* selection function γ. To show $\star = \star_{fm}$ we show that, for the selection function γ constructed in the completeness proof, the additional postulate **(FM)** actually forces $\gamma(u \perp (\varphi/a)) = u \perp (\varphi/a)$ in the case $u \perp (\varphi/a) \neq \emptyset$, i.e., that $u \sqcap (u \star (\varphi/a)) \sqsubseteq u'$ for **all** $u' \in u \perp (\varphi/a)$. So let $u' \in u \perp (\varphi/a)$. If it were the case that $u \sqcap (u \star (\varphi/a)) \not\sqsubseteq u'$ then we could use the properties of $u \perp (\varphi/a)$ to deduce $u' \sqcup (u \sqcap (u \star (\varphi/a))) \sqcup (\varphi/a)$ is inconsistent and hence, by **Con**$\downarrow$, $u' \sqcup (u \star (\varphi/a)) \sqcup (\varphi/a)$ is inconsistent. But this last line contradicts the fact that $\star$ satisfies **(FM)**, as can be seen by setting $v = u' \sqcup (\varphi/a)$ in that rule. Hence we must have $u \sqcap (u \star (\varphi/a)) \sqsubseteq u'$ as required. ∎

Full meet fuzzy base revision is a very cautious form of revision. As has been pointed out in the crisp case (see [17]), it is not suitable as a general definition of revision since it leads to *too much* information being given up from the old base to accommodate the revision input.

5.2. Maxichoice fuzzy base revision

At the other end of the spectrum, we have maxichoice fuzzy base revision. $\star$ is a maxichoice fuzzy base revision operator iff $\star$ is a partial meet fuzzy base revision operator for which the selection function γ selects only a *single element* of $u \perp (\varphi/a)$. (For the case $u \perp (\varphi/a) = \emptyset$ we have $\gamma(u \perp (\varphi/a)) = \{u\}$ as usual.) The following postulate will help to characterise maxichoice fuzzy base revision.

(MC) For all $\theta \in L$, $b \in W$, if $b \not\leq_W [u \star (\varphi/a)](\theta)$ and $b \leq_W u(\theta)$ then $(u \star (\varphi/a)) \sqcup (\theta/b)$ is inconsistent.

(MC) is essentially a generalisation of the postulate "Tenacity" from crisp base revision [7, 15]. As is discussed in [17, p209], it is unreasonable as a general property of revision since it causes beliefs which are given up during a revision to swing unrealistically from being accepted to at once becoming inconsistent with the revised base. We get the following characterisation of maxichoice fuzzy base revision, which essentially generalises the one for maxichoice crisp base revision given in [17]:

THEOREM 5.2. *Let $u \in \mathcal{F}(L)$ and let $\star$ be an operator for u. Then $\star$ is an operator of maxichoice fuzzy base revision iff $\star$ satisfies* **(F1)**–**(F5)** *and* **(MC)**.

PROOF (*Outline*). Soundness: Since every maxichoice fuzzy base revision operator is a partial meet fuzzy base revision operator we know already from Theorem 4.1 that **(F1)**–**(F5)** are satisfied, so it remains to show **(MC)**. So let $\star$ be a maxichoice fuzzy base revision operator, i.e., $u \star (\varphi/a) = \gamma(u \perp (\varphi/a)) \sqcup (\varphi/a)$ where $\gamma(u \perp (\varphi/a))$ is an element of $u \perp (\varphi/a)$ if $u \perp (\varphi/a) \neq \emptyset$[13], and $\gamma(u \perp (\varphi/a)) = u$ if $u \perp (\varphi/a) = \emptyset$. Suppose θ and b are such that $b \not\leq_W [u \star (\varphi/a)](\theta)$, i.e., $b \not\leq_W [\gamma(u \perp (\varphi/a)) \sqcup (\varphi/a)](\theta)$, and $b \leq_W u(\theta)$. We must show $\gamma(u \perp (\varphi/a)) \sqcup (\varphi/a) \sqcup (\theta/b)$ is inconsistent. Note the first assumption on θ, b implies $b \not\leq_W [\gamma(u \perp (\varphi/a))](\theta)$, so it clearly cannot be the case that $\gamma(u \perp (\varphi/a)) = u$. Hence we must have $\gamma(u \perp (\varphi/a)) \in u \perp (\varphi/a)$. Hence, using the properties of $u \perp (\varphi/a)$ it suffices to show $\gamma(u \perp (\varphi/a)) \sqsubset \gamma(u \perp (\varphi/a)) \sqcup (\theta/b) \sqsubseteq u$. The first, strict inequality follows from $b \not\leq_W [\gamma(u \perp (\varphi/a))](\theta)$, while the second follows with the help of $b \leq_W u(\theta)$.

Completeness: We show that the addition of **(MC)** to **(F1)**–**(F5)** forces the selection function γ constructed in the completeness proof of Theorem 4.1 to always select a singleton subset of $u \perp (\varphi/a)$ in the case $u \perp (\varphi/a) \neq \emptyset$. In fact we show that, in this case, $u \sqcap (u \star (\varphi/a)) \in u \perp (\varphi/a)$. Then since, as is easily checked, we have that for all $x, y \in u \perp (\varphi/a)$, $x \sqsubseteq y$ implies $x = y$, we get $\gamma(u \perp (\varphi/a)) = \{u \sqcap (u \star (\varphi/a))\}$ and so $\gamma(u \perp (\varphi/a))$ is a singleton as required. From the completeness proof of Theorem 4.1 we already know $u \sqcap (u \star (\varphi/a))$ satisfies conditions (i) and (ii) from Definition 3.1. It remains to prove condition (iii). So let $u \sqcap (u \star (\varphi/a)) \sqsubset u' \sqsubseteq u$. We must show $u' \sqcup (\varphi/a)$ is inconsistent. But these inequalities together imply $u' \not\sqsubseteq u \star (\varphi/a)$,

[13] For the sake of this proof's readability we slightly abuse notation here: $\gamma(u \perp (\varphi/a))$ is, of course, really a singleton set *containing* the selected element of $u \perp (\varphi/a)$.

so there exists $\theta \in L$ such that $u'(\theta) \not\leq_W [u \star (\varphi/a)](\theta)$. Remembering also $u' \sqsubseteq u$ we may apply **(MC)** (substituting $u'(\theta)$ for b there) to deduce $(u \star (\varphi/a)) \sqcup (\theta/u'(\theta))$ is inconsistent. Now from $u \sqcap (u \star (\varphi/a)) \sqsubseteq u'$ we get $(u \sqcap (u \star (\varphi/a))) \sqcup (\varphi/a) \sqsubseteq u' \sqcup (\varphi/a)$. Using the distributivity of $\mathcal{F}(L)$ followed by applications of **(F1)** and **(F3)**, we see the left-hand side of this last inequality is equal to $u \star (\varphi/a)$. Hence $u \star (\varphi/a) \sqsubseteq u' \sqcup (\varphi/a)$ and so $(u \star (\varphi/a)) \sqcup (\theta/u'(\theta)) \sqsubseteq u' \sqcup (\varphi/a)$. Hence $u' \sqcup (\varphi/a)$ is inconsistent by **Con**$\downarrow$ as required. ■

Note that we could replace **(F4)** in the above characterisation by the simpler **(F9)**, since it can be shown that, in the presence of **(F2)** and **(F3)**, **(F4)** follows from **(MC)** and **(F9)**. [Hint: in **(F4)**, put $u' = u \star (\varphi/a)$ and note that, by **(F9)**, θ, b cannot exist if (φ/a) is inconsistent.]

6. Partial meet fuzzy base contraction

As described at the beginning of Section 3, in the special crisp case the AGM view of revision (which actually traces back to Levi [19]) is as a composite operation made up of two sub-operations: contraction followed by expansion. Expansion is a trivial operation which merely consists in joining the new information to the contracted base. Clearly, all the dirty work here is left to the contraction operation itself, whose job is to weaken the initial base to make it consistent with the new information. Naturally, one could concentrate on contraction as the primary operation of interest, with revision then seen as a merely derived operation, and indeed this path is commonly taken in the literature on crisp belief change. For our part, instead of partial meet fuzzy base *revision*, we could just as easily have started with an operation of partial meet fuzzy base *contraction*, formed by following only the first three steps of the procedure given on page 38. Precisely, given $u \in \mathcal{F}(L)$ and a selection function γ for u we define the operator $\ominus_\gamma$ for u by setting, for any $(\varphi/a) \in L \times W$,

$$u \ominus_\gamma (\varphi/a) = \bigsqcap \gamma(u \perp (\varphi/a)).$$

Definition 6.1. Let $u \in \mathcal{F}(L)$. Then $\ominus : L \times W \to \mathcal{F}(L)$ is an operator of *partial meet fuzzy base contraction* (for u) iff $\ominus = \ominus_\gamma$ for some selection function γ for u.

Obviously, for any selection function γ, the partial meet fuzzy base revision operator $\star_\gamma$ can be defined in terms of the corresponding $\ominus_\gamma$ in the following way:

$$u \star_\gamma (\varphi/a) = (u \ominus_\gamma (\varphi/a)) \sqcup (\varphi/a). \tag{Levi}$$

Less obviously, Proposition 3.8 tells us how we can define $\ominus_\gamma$ in terms of $\star_\gamma$:

$$u \ominus_\gamma (\varphi/a) = u \sqcap (u \star_\gamma (\varphi/a)). \qquad \text{(Harper)}$$

In the case of crisp deduction systems, partial meet fuzzy base contraction reduces to the operation of partial meet base contraction given in [1, 14].[14] As with partial meet fuzzy base revision, it turns out we may give these operators an axiomatisation which generalises the one obtained in the crisp case (see [14]). Again the postulate names on the right correspond to the ones used for their instances in the crisp case.

THEOREM 6.2. *Let $u \in \mathcal{F}(L)$ and $\ominus$ be an operator for u. Then $\ominus$ is an operator of partial meet fuzzy base contraction for u iff $\ominus$ satisfies:*

(G1) $(u \ominus (\varphi/a)) \sqcup (\varphi/a)$ *is consistent if* (φ/a) *is consistent* *(Success)*

(G2) $u \ominus (\varphi/a) \sqsubseteq u$ *(Inclusion)*

(G3) *For all $\theta \in L$, $b \in W$, if $b \not\leq_W [u \ominus (\varphi/a)](\theta)$ and $b \leq_W u(\theta)$ then there exists u' such that $u \ominus (\varphi/a) \sqsubseteq u' \sqsubseteq u$, $u' \sqcup (\varphi/a)$ is consistent and $u' \sqcup (\varphi/a) \sqcup (\theta/b)$ is inconsistent.* *(Relevance)*

(G4) *If, for all $x \sqsubseteq u$, we have $x \sqcup (\varphi/a)$ is consistent iff $x \sqcup (\varphi'/a')$ is consistent, then $u \ominus (\varphi/a) = u \ominus (\varphi'/a')$.* *(Uniformity)*

PROOF (*Outline*). Soundness: We use (Levi) and (Harper) above and the properties already proved for $\star_\gamma$. **(G1)** follows immediately from (Levi) and the fact that $\star_\gamma$ satisfies **(F1)**. **(G2)** is obvious. **(G3)** can be proved using the fact that **(F4)** holds. To see this let θ, b be such that $b \not\leq_W [u \ominus_\gamma (\varphi/a)](\theta)$ and $b \leq_W u(\theta)$. The former is equivalent to $b \not\leq_W [u \sqcap (u \star_\gamma (\varphi/a))](\theta)$, i.e., $b \not\leq_W u(\theta) \curlywedge [u \star_\gamma (\varphi/a)](\theta)$. Since we assume $b \leq_W u(\theta)$, we must have $b \not\leq_W [u \star_\gamma (\varphi/a)](\theta)$. We may now use the fact that $\star_\gamma$ satisfies **(F4)** and deduce the existence of some u'' such that $u \star_\gamma (\varphi/a) \sqsubseteq u'' \sqsubseteq u \sqcup (\varphi/a)$, u'' is consistent and $u'' \sqcup (\theta/b)$ is inconsistent. It can then be shown that setting $u' = u'' \sqcap u$ gives us the required u' in **(G3)**. Finally **(G4)** follows immediately from (Harper) and the fact that $\star_\gamma$ satisfies **(F5)**.

Completeness: We show that for any operator $\ominus$ satisfying **(G1)**–**(G4)** there exists some partial meet fuzzy base revision operator $\star$ for u such that $u \ominus (\varphi/a) = u \sqcap (u \star (\varphi/a))$. This suffices since if we can choose some selection function γ such that $\star = \star_\gamma$ here, then we get $\ominus = \ominus_\gamma$ by (Harper).

[14] Recall that, in this case, making u consistent with φ amounts to ensuring $\neg\varphi$ cannot be deduced from u.

To find this $\star$ we simply use the Levi recipe and define $\star'$ from $\ominus$ by setting $u \star' (\varphi/a) = (u \ominus (\varphi/a)) \sqcup (\varphi/a)$. That $u \ominus (\varphi/a) = u \sqcap (u \star' (\varphi/a))$ holds since $u \sqcap (u \star' (\varphi/a)) = u \sqcap ((u \ominus (\varphi/a)) \sqcup (\varphi/a)) = (u \sqcap (u \ominus (\varphi/a)) \sqcup (u \sqcap (\varphi/a)) = (u \ominus (\varphi/a)) \sqcup (u \sqcap (\varphi/a))$ using **(G2)**. This latter is then equal to the required $u \ominus (\varphi/a)$, since it can be shown that the property "$u \sqcap (\varphi/a) \sqsubseteq u \ominus (\varphi/a)$" is derivable from **(G3)**. [Hint: this clearly boils down to showing $u(\varphi) \curlywedge a \leq_W [u \ominus (\varphi/a)](\varphi)$. To see this holds try setting $\theta = \varphi$ and $b = u(\varphi) \curlywedge a$ in **(G3)**.] We show $\star'$ is a partial meet fuzzy base revision operator by checking each of **(F1)**–**(F5)** are satisfied. **(F1)** is obvious while **(F2)**–**(F4)** follow since $\ominus$ satisfies **(G1)**–**(G3)** respectively. Finally **(F5)** is valid by the fact that $\ominus$ satisfies **(G4)**, together with the just proven identity $u \ominus (\varphi/a) = u \sqcap (u \star' (\varphi/a))$. ■

7. A note on the generality of our results

Before concluding, we would like to point out something regarding the generality of the results described in Sections 3–6. As we indicated in our question at the beginning of Section 3, the results apply with any arbitrary abstract fuzzy deduction system (L, W, D) as background. This means that D is always assumed to be a logically compact fuzzy deduction operator which satisfies the properties Monotony, Idempotence and Reflexivity. As we have seen in our examples, this already means that the results apply to quite a wide variety of cases, from crisp deduction systems, through truth-functional fuzzy logics, to probabilistic logics. However, a close inspection of our proofs reveals that the only properties of D which are actually *used* to prove the results are logical compactness and **Con**$\downarrow$.[15] (The single exception to this is the derivation of rule **(F10)** in Proposition 4.2, which requires all three of the (generalised) Tarski properties.) These two properties have in common that they are, first-and-foremost, constraints on the set of D*-consistent* fuzzy bases, rather than on D itself. A closer look at the sketched procedure on page 38 will reveal why this is no coincidence, for we have presented partial meet revision (and contraction) as a wholly "consistency-driven" process – the operator D does not enter the discussion in any way beyond its associated notion of consistency. This means that in fact the formal results described are potentially applicable in an even wider variety of situations than the ones we've mentioned here, since all that is needed to get going, along with

[15] We remark that, for the crisp case, it is already noticed in [18, Section 3] that the only properties required of D are Compactness and Monotony (which, in the presence of Deduction, is actually equivalent to **Con**$\downarrow$).

the set L of formulas and the complete, distributive lattice W of degrees, is some well-behaved (i.e., satisfying logical compactness and **Con**$\downarrow$) notion of when a fuzzy base is consistent.

8. Conclusion

We have considered the question of fuzzy belief base revision within Gerla's general framework for fuzzy logic. We have defined and axiomatised the operation of partial meet fuzzy base revision, which generalises the operation of partial meet base revision from the usual crisp case. The fact that we obtained this axiomatisation with such relatively weak restrictions shows on the one hand how the ideas of rational belief change are general enough to be applied to reasoning under vagueness or uncertainty. On the other hand, it confirms that the types of fuzzy systems covered by our abstract setting are indeed appropriate for modelling the human capacity of making conclusions from uncertain or vague premises. We have given some examples which show how the operation works in some specific instances of the framework, including those related to Łukasiewicz fuzzy logic and probability logic. We have also axiomatised the corresponding operation of partial meet fuzzy base contraction.

In this paper the question of base revision has been investigated from a very high position on the abstraction ladder, with only a handful of properties assumed of the basic primitives. We have shown that it is nevertheless possible to formulate basic properties of base revision operators. We would like to think of **(F1)**–**(F5)** as the absolute minimal *core* properties which any base revision operator should satisfy. However, as we move down the abstraction ladder, we fully expect to be able to say more. Furthermore, as the differences between the various instantiations of our abstract framework then come into focus, such as those between truth-functional logic and uncertainty calculi (e.g. probability logic), we also expect to be able to answer another important question: are there postulates suitable for revision in one setting which are unsuitable in another? This will be left for future work, as will the consideration of postulates which govern the revision of a base by different, but related inputs. What, for example (assuming we work in L_{Prop}), is the connection between $u \star (\varphi/b)$ and $u \star (\theta \wedge \varphi/b)$? Also in this category would be some property of *robustness*, i.e., the idea that small changes in the degree a of the revision input (φ/a) should cause only small changes to $u \star (\varphi/a)$ (particularly relevant if $W = [0, 1]$). Probably the fulfillment of conditions like these by partial meet fuzzy base revision operators will require some restriction on the selection function γ. Some preliminary investigations into the latter suggest we get robustness if we additionally restrict to continuous truth-functional semantics. Finally we would also like to study *theory* revision in this framework.

Acknowledgements

R.B.'s work is supported by the DFG project "Computationale Dialektik". E.R.'s work was supported by the DFG doctoral program "Wissensrepräsentation" at the University of Leipzig. Thanks are due to Jeff Paris and Jacek Malinowski for some useful comments and suggestions, and to the referees, both of the UAI'03 conference and of this journal, whose comments helped to improve the paper.

Appendix: A syntactic characterisation of D_{P}

In this appendix we sketch a syntactic characterisation of the probabilistic deduction operator D_{P} which we introduced in Section 2.1.4. This characterisation, which takes the form of a proof system, was presented originally in [8, 9], and for a more detailed treatment we refer the reader to those papers. To explain the result here, it is helpful to first describe a general framework of *fuzzy proof systems* which was first given in [21]. The proof system for D_{P} is then obtained as a particular instantiation of this framework.

Starting from any set L of formulas and set W of degrees, a fuzzy proof system for $\mathcal{F}(L)$ is a pair $\mathcal{S} = (Lax, R)$. As with classical proof systems, this consists of a set of logical axioms Lax and a set of rules of inference R. The difference is that $Lax \in \mathcal{F}(L)$ is now a fuzzy base and the rules in R are now *fuzzy rules of inference*. Such rules $r = (r', r'')$ consist of two parts: an *L-part* r' and a *W-part* r''. The former is a function which takes as input a tuple $(\varphi_1, \ldots, \varphi_{n(r)}) \in Dom(r) \subseteq L^{n(r)}$, (where $n(r)$ is the *arity* of the rule r) and returns a formula in L. The set $Dom(r)$ demarcates the *domain of application* of the rule r. The W-part is a function $r'' : W^{n(r)} \to W$, such that $r''(a_1, \ldots, sup_{i\in I} b_i, \ldots a_{n(r)}) = sup_{i \in I} r''(a_1, \ldots, b_i, \ldots, a_{n(r)})$. The informal interpretation of such a fuzzy rule of inference r is: if $(\varphi_1, \ldots, \varphi_{n(r)}) \in Dom(r)$ and we have derived that the degree of each φ_i is at least a_i, then we may deduce that the degree of $r'(\varphi_1, \ldots, \varphi_{n(r)})$ is at least $r''(a_1, \ldots, a_{n(r)})$. For $\varphi \in L$, an *$\mathcal{S}$-proof* of φ is then a finite sequence $\pi = \varphi_1, \ldots, \varphi_k$ of formulas with $\varphi = \varphi_k$, such that for each i, either *(i)* φ_i is declared as a logical axiom, *(ii)* φ_i is declared as a non-logical axiom, or *(iii)* there exists a fuzzy inference rule r with $r'(\varphi_{j_1}, \ldots, \varphi_{j_{n(r)}}) = \varphi_i$ for some formulas $(\varphi_{j_1}, \ldots, \varphi_{j_{n(r)}}) \in Dom(r)$ where each $j_l < i$. Given a fuzzy base $u \in \mathcal{F}(L)$

as an initial valuation the *valuation* $Val(\pi, u)$ of an $\mathcal{S}$-proof π *with respect to* u is defined by induction on the length k of π by setting:

$$Val(\pi, u) = \begin{cases} Lax(\varphi_k) & \text{if } \varphi_k \text{ is declared as a logical axiom,} \\ u(\varphi_k) & \text{if } \varphi_k \text{ is declared as a non-logical axiom,} \\ r''(Val(\pi(j_1), u), \ldots Val(\pi(j_{n(r)}), u)) & \\ & \text{if } \varphi_k = r'(\varphi_{j_1}, \ldots \varphi_{j_{n(r)}}). \end{cases}$$

where $\pi(j)$ is the proof formed by taking just the first j elements of π. Each $\mathcal{S}$ then yields a fuzzy deduction operator $D_{\mathcal{S}}$ by setting, for each $u \in \mathcal{F}(L)$ and $\varphi \in L$,

$$D_{\mathcal{S}}(u)(\varphi) = sup\{Val(\pi, u) \mid \pi \text{ is an } \mathcal{S}\text{-proof for } \varphi\}.$$

The reader may be warned here that logical compactness of $D_{\mathcal{S}}$ does not imply the compactness of the notion of proof, i.e., for an arbitrary formula φ in general there does not exist a "maximal" proof π_{max} with $Val(\pi_{max}, u) = D_{\mathcal{S}}(u)(\varphi)$.

Now, for the probabilistic case we take $L = L_{\mathrm{Prop}}$, $W = [0, 1]$. For the logical axioms we take the fuzzy base Lax_{P} defined by $Lax_{\mathrm{P}}(\varphi) = 1$ if $\varphi \in Cn(\emptyset)$, $Lax_{\mathrm{P}}(\varphi) = 0$ otherwise. To specify our fuzzy inference rules, we require some extra notation. Letting $(L_{\mathrm{Prop}})^*$ denote the set of all finite non-empty sequences of formulas, we define, for each $k \in \mathbb{N}$ a function $C^k : (L_{\mathrm{Prop}})^* \to L_{\mathrm{Prop}}$ by setting, for each $(\varphi_1, \ldots, \varphi_h) \in (L_{\mathrm{Prop}})^*$,

$$C^k(\varphi_1, \ldots, \varphi_h) = \begin{cases} \top & \text{if } k = 0, \\ \bigvee\{\varphi_{i_1} \wedge \ldots \wedge \varphi_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\} & \text{otherwise} \end{cases}$$

where $\top$ is some fixed classical tautology. (In view of the Cn-equivalence inference rules defined below, the precise order of the disjuncts in the second clause here is unimportant.) In other words $C^k(\varphi_1, \ldots, \varphi_h)$ returns the disjunction of all conjunctions of k elements from $(\varphi_1, \ldots, \varphi_h)$ in the case $k \neq 0$. Note if $k > h$ above then $C^k(\varphi_1, \ldots, \varphi_h)$ is an empty disjunction, which by convention we take to be a classical contradiction. We also define a function $M : (L_{\mathrm{Prop}})^* \to \mathbb{N}$ by setting $M(\varphi_1, \ldots, \varphi_h) = \max\{k \in \mathbb{N} \mid C^k(\varphi_1, \ldots, \varphi_h) \text{ is } Cn\text{-consistent}\}$.

With the help of these functions we now define three types of fuzzy inference rules.

- The *h-m-k-rules* are all the rules $r = (r', r'')$ of the form:

$$r'(\varphi_1, \ldots, \varphi_h) = C^k(\varphi_1, \ldots, \varphi_h); \quad r''(a_1, \ldots, a_h) = [\tfrac{a_1 + \cdots + a_h - k + 1}{m - k + 1}]$$

for each $h \geq m \geq k$. Here $Dom(r) = \{(\varphi_1, \dots, \varphi_h) \mid M(\varphi_1, \dots, \varphi_h) = m\}$ and the function $[\cdot] : \mathbb{R} \to [0, 1]$ is defined by $[s] = 0$ if $s < 0$, $[s] = 1$ if $s > 1$, and $[s] = s$ otherwise.

- The *h-m-collapsing rules* are all rules $c = (c', c'')$ of the form:

$$\begin{aligned} c'(\varphi_1, \dots, \varphi_h) &= \bot \text{ where } \bot \text{ is some fixed classical contradiction} \\ c''(a_1, \dots, a_h) &= \begin{cases} 1 & \text{if } a_1 + \dots + a_h > m, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

  for each $h \geq m$ with again $Dom(c) = \{(\varphi_1, \dots, \varphi_h) \mid M(\varphi_1, \dots, \varphi_h) = m\}$. For any base u we have u is D_{P}-consistent iff $u(\varphi_1) + \dots + u(\varphi_h) \leq M(\varphi_1, \dots, \varphi_h)$ for all φ_i (see [8, 9]). Hence the role of these rules is to deduce a classical contradiction (and thus in fact all formulas) with degree 1 as soon as a D_{P}-inconsistency is detected.

- Finally the *Cn-equivalence rules* are all rules $s = (s', s'')$ of the form: $s'(\varphi) = \psi; s''(a) = a$, where $(\varphi \leftrightarrow \psi) \in Cn(\emptyset)$ (and $Dom(s) = L_{\mathrm{Prop}}$). These rules enable us to freely replace any formula in a proof by a logically equivalent one. They were not needed in [9], since the author considers the Lindenbaum algebra rather than L_{Prop}.

Letting $\mathcal{S}_{\mathrm{P}}$ denote the fuzzy proof system $(Lax_{\mathrm{P}}, R_{\mathrm{P}})$, with R_{P} consisting of all rules of all three types above, we then have the following result:

THEOREM A.1 ([9]). *$D_{\mathrm{P}} = D_{\mathcal{S}_{\mathrm{P}}}$.*

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