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Belief Liberation (and Retraction)

Abstract. We provide a formal study of belief retraction operators that do not necessarily satisfy the (Inclusion) postulate. Our intuition is that a rational description of belief change must do justice to cases in which dropping a belief can lead to the inclusion, or 'liberation', of others in an agent's corpus. We provide two models of liberation via retraction operators: σ-liberation and linear liberation. We show that the class of σ-liberation operators is included in the class of linear ones and provide axiomatic characterisations for each class. We show how any retraction operator (including the liberation operators) can be 'converted' into either a withdrawal operator (i.e., satisfying (Inclusion)) or a revision operator via (a slight variant of) the Harper Identity and the Levi Identity respectively.

Keywords: Belief revision, AGM theory, contraction, withdrawal, Inclusion postulate.

1. Introduction

Formal modelings of rational belief change are inevitably interested in plausible descriptions of the process of dropping beliefs. The *AGM framework*, named after its originators Alchourrón, Gärdenfors and Makinson [1, 8], characterises belief contraction via a set of postulates. One of these, (Inclusion), states that the belief set that is the result of contraction must be included in the belief set prior to contraction. Justifications for (Inclusion) are hard to find – it is usually just taken for granted. But, there are situations in which *the removal of a belief might lead to the inclusion of new ones.* Consider an agent that keeps track of information received and which has received both $\neg\phi$ and then ϕ over a period of time. When it draws inferences from this set of information, it prioritises more recent information and hence does not infer $\neg\phi$. But information that causes it to retract ϕ can be viewed as also leading to either an increase in the plausibility of $\neg\phi$ or even to a belief in $\neg\phi$ and other beliefs that were blocked by ϕ. A similar situation occurs in settings involving default reasoning [16]. If an agent was committed to a default rule that sanctioned belief in ϕ provided it was consistent to assume ψ, and also believed $\neg\psi$ to be true, it would be unable to

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apply the default rule and would consequently not believe ϕ. Retraction of the belief $\neg\psi$ makes the default rule applicable, thus sanctioning belief in ϕ.

We believe that the overriding messages from examples like these is that removing one belief might remove the grounds for withholding another. That is, when a 'blocking' belief is removed from an agent's belief corpus, so are the reasons or arguments against other beliefs which the agent had not previously entertained. Such a model is in the spirit of a *foundational* approach to belief change [5] and this is as it should be, since an agent's corpus is most plausibly viewed as a set of beliefs along with the reasons for holding them. Thus, belief retractions can be 'liberating': beliefs which were blocked are 'set free'. In this paper, we start from a set of basic postulates for *retraction* which *excludes* (Inclusion) and also the much-debated postulate of (Recovery). The broad class of operators so defined is designed to include the 'traditional' operators of AGM contraction and *withdrawal* [13], but our main focus is to study those retractions which can be viewed as liberation operators. We do not aim to jettison the Principle of Minimal Change in this study – the intuitions there are certainly worth retaining. Doing justice to that particular methodological principle – while not ignoring other equally important ones[1] – and rejecting (Inclusion) will be an objective of ours. A formal argument which supports our pre-theoretic intuitions is that it is well-known that when defining a *revision* operator $*$ from an AGM contraction operator $\div$ via the Levi Identity ([8], see also Sect. 5.2 of this paper), $\div$ isn't required to satisfy (Recovery) to ensure $*$ satisfies the AGM revision postulates. Less widely acknowledged is the fact that $\div$ doesn't have to satisfy (Inclusion) either. That is, if $\div$ is a retraction and $*$ is defined from $\div$ via Levi then $*$ is a partial meet revision.

We begin in Sect. 2 by formally defining retraction operators. In Sect. 3 we provide two models of liberation via retraction operators: *σ-liberation* and *linear liberation*. Each of these utilises a finite *sequence* of sentences which guides the operation of belief removal. Though they differ in the *way* they utilise the sequence, we will show that the class of σ-liberation operators is included in the class of linear liberation operators and provide axiomatic characterisations for each class. We also axiomatise a number of subclasses of linear liberation. Sect. 4 is devoted to some weaker versions of the (Inclusion) postulate. In Sect. 5 we show how a given retraction operator can be 'converted' into either a withdrawal operator (satisfying (Inclusion)) or a revision operator using (a slight variant of) the Harper Identity and the

[1] Such an approach is explicit in the work of Rott and Pagnucco[17], and Meyer et al.[14] where the Principle of Minimal Change gives way to other methodological principles.

Levi Identity respectively. We briefly conclude in Sect. 6 before finishing off with some ideas for further work in Sect. 7.

We assume a propositional language L generated by finitely many propositional variables. We use $\models$ to denote classical entailment and Cn to denote the classical logical consequence operator; $\top, \bot$ have their usual meanings. We assume that the object of change is a *consistent* belief set K i.e., a deductively closed set of sentences. While this is not always an explicit assumption, it is almost always intended to be the case. We take K to be arbitrary and fixed throughout. As is usual we use $K + \phi$ to denote $Cn(K \cup \{\phi\})$. The set of propositional models of K will be denoted by $M(K)$.

2. Postulates for retraction

We first present the *basic AGM postulates*, which characterise *partial meet contraction* [1]. We use $K \ominus \phi$ to denote the result of removing the sentence ϕ from K.

(L1) $K \ominus \phi = Cn(K \ominus \phi)$ (Closure)

(L2) If $\not\models \phi$ then $\phi \notin K \ominus \phi$ (Success)

(L3) If $\phi \notin K$ then $K \ominus \phi = K$ (Vacuity)

(L4) If $\models \phi_1 \leftrightarrow \phi_2$ then $K \ominus \phi_1 = K \ominus \phi_2$ (Extensionality)

(L5) $K \ominus \phi \subseteq K$ (Inclusion)

(L6) $K \subseteq (K \ominus \phi) + \phi$ (Recovery)

(Recovery) has already been seen as problematic (e.g., [9, 13]). Following [13], we call any operator which satisfies **(L1)**–**(L5)** a *withdrawal* operator. We want now to go a step further and shed (Inclusion) as well. However we keep the following basic condition, which follows from **(L1)**, **(L5)** and **(L6)**:

$$K \ominus \top = K \qquad \text{(Failure)[7]}$$

Definition 2.1. *Let K be a belief set and $\ominus$ be an operator for K. Then $\ominus$ is a* retraction operator (for K) *iff $\ominus$ satisfies* **(L1)**–**(L4)** *and (Failure).*

3. Models of liberation

We now present two models of liberation operators; each will be presented in terms of finite sequences of sentences. The class of liberation operators generated by the second includes that generated by the first.

3.1. σ-liberation

In our first model, the central intuition is that both the agent's set of beliefs and the way it removes beliefs are formed on the basis of the information that it has received over the course of its intellectual career. We assume the agent has at its disposal a finite *belief sequence* $\sigma = (\alpha_1, \ldots, \alpha_n)$ of sentences, with α_n being the most recent information the agent has received[2]. What beliefs is the agent committed to on the basis of σ, i.e., what is the belief set K_σ *associated with* σ? An obvious answer would be to take the set $[\![\sigma]\!]$ of all the sentences appearing in σ and to then close under Cn. The problem with this answer, of course, is that we would like K_σ to be consistent, and it could well be that $[\![\sigma]\!]$ is *inconsistent*. Instead we use the priority of information encoded in σ to help us – initially – pick out consistent subsets of $[\![\sigma]\!]$. We define the increasing sequence of sets $\Gamma_i(\sigma)$ inductively by setting $\Gamma_0(\sigma) = \emptyset$ and then, for each $i = 0, 1, \ldots, n-1$,

$$\Gamma_{i+1}(\sigma) = \begin{cases} \Gamma_i(\sigma) \cup \{\alpha_{n-i}\} & \text{if } \Gamma_i(\sigma) \cup \{\alpha_{n-i}\} \not\models \bot \\ \Gamma_i(\sigma) & \text{otherwise} \end{cases}$$

That is, starting with α_n, we work our way backwards through the sequence, adding each sentence as we go, provided it is consistent with the sentences collected up to that point. We then take $Cn(\Gamma_n(\sigma))$ to be the belief set associated with σ.

DEFINITION 3.1. *Let K be a belief set and $\sigma = (\alpha_1, \ldots, \alpha_n)$ a belief sequence. We say σ is a belief sequence* relative to K *iff $K = Cn(\Gamma_n(\sigma))$.*

EXAMPLE 3.2. Suppose $\sigma = (\neg p \wedge \neg q, p, p \rightarrow q)$ where p and q are distinct propositional variables. Then $\Gamma_0(\sigma) = \emptyset$, $\Gamma_1(\sigma) = \{p \rightarrow q\}$, $\Gamma_2(\sigma) = \{p, p \rightarrow q\} = \Gamma_3(\sigma)$. Hence the belief set K associated with this σ is given by $K = Cn(\Gamma_3(\sigma)) = Cn(p \wedge q)$. Note how belief in the first/oldest sentence $\neg p \wedge \neg q$ in σ is suppressed in particular by the more recent sentence p.

Given a belief sequence σ relative to K, we want to use σ to define an operation $\ominus_\sigma$ for K such that $K \ominus_\sigma \phi$ represents the result of removing ϕ from K. If ϕ is a tautology we just set $K \ominus_\sigma \phi = K$. Otherwise we introduce sequences of sets $\Gamma_i(\sigma, \phi)$ inductively by setting $\Gamma_0(\sigma, \phi) = \emptyset$ and then, for each $i = 0, 1, \ldots, n-1$,

$$\Gamma_{i+1}(\sigma, \phi) = \begin{cases} \Gamma_i(\sigma, \phi) \cup \{\alpha_{n-i}\} & \text{if } \Gamma_i(\sigma, \phi) \cup \{\alpha_{n-i}\} \not\models \phi \\ \Gamma_i(\sigma, \phi) & \text{otherwise} \end{cases}$$

[2]The sentences can stand for anything, not just a record of observations. The main thing is that we have a linearly ordered/prioritised set of sentences. Such a treatment is reminiscent of [3]. See also [15].

That is, starting at the end with α_n, we work our way backwards through the sequence, adding each sentence as we go, provided adding it to the sentences collected up to that point does not lead to the inference of ϕ. Note that $\Gamma_i(\sigma) = \Gamma_i(\sigma, \bot)$. We then set

$$K \Bumpeq_\sigma \phi = \begin{cases} Cn(\Gamma_n(\sigma, \phi)) & \text{if } \not\models \phi \\ K & \text{otherwise} \end{cases}$$

DEFINITION 3.3. *Let K be a belief set and $\Bumpeq$ be an operator for K. Then $\Bumpeq$ is a σ-*liberation *operator (for K) iff $\Bumpeq = \Bumpeq_\sigma$ for some belief sequence σ relative to K.*

EXAMPLE 3.4. Suppose $K = Cn(p \wedge q)$ and let σ from Example 3.2 be the belief sequence relative to K. Suppose we wish to remove p. We first compute $\Gamma_3(\sigma, p)$. We have $\Gamma_0(\sigma, p) = \emptyset$, $\Gamma_1(\sigma, p) = \{p \rightarrow q\} = \Gamma_2(\sigma, p)$ and $\Gamma_3(\sigma, p) = \{\neg p \wedge \neg q, p \rightarrow q\}$. Hence $K \Bumpeq_\sigma p = Cn(\Gamma_3(\sigma, p)) = Cn(\neg p \wedge \neg q)$. Note how, at the second stage, p is *nullified*, which leads to the reinstatement, or liberation, of $\neg p \wedge \neg q$.

As the above example shows, σ-liberation operators do not necessarily satisfy (Inclusion). What properties *are* satisfied by σ-liberation? Well, first of all, we can confirm that σ-liberation is indeed a retraction operator according to our basic definition:

PROPOSITION 3.5. *Every σ-liberation operator satisfies the basic retraction postulates —* **(L1)**–**(L4)** *and (Failure) — and so is a retraction operator.*

PROOF. Let $\sigma = (\alpha_1, \dots, \alpha_n)$ be a belief sequence relative to K. We check that $\Bumpeq_\sigma$ satisfies each postulate in turn.
(L1) $K \Bumpeq_\sigma \phi = Cn(K \Bumpeq_\sigma \phi)$ If $\models \phi$ then $K \Bumpeq_\sigma \phi = K$ and the rule holds since K is a belief set and so $K = Cn(K)$. If $\not\models \phi$ then $K \Bumpeq_\sigma \phi = Cn(\Gamma_n(\sigma, \phi))$ and the rule holds by the idempotence of Cn.
(L2) If $\not\models \phi$ then $\phi \notin K \Bumpeq_\sigma \phi$ If $\not\models \phi$ then $K \Bumpeq_\sigma \phi = Cn(\Gamma_n(\sigma, \phi))$. By an easy inductive proof $\phi \notin Cn(\Gamma_i(\sigma, \phi))$ for all $i = 0, 1, \dots, n$. In particular $\phi \notin Cn(\Gamma_n(\sigma, \phi))$.
(L3) If $\phi \notin K$ then $K \Bumpeq_\sigma \phi = K$ Suppose $\phi \notin K$. Then $\not\models \phi$ and so $K \Bumpeq_\sigma \phi = Cn(\Gamma_n(\sigma, \phi))$. Meanwhile $K = Cn(\Gamma_n(\sigma))$, and so $\phi \notin Cn(\Gamma_n(\sigma))$. We will show, by induction on i, that $\Gamma_i(\sigma, \phi) = \Gamma_i(\sigma)$ for all $i = 0, 1, \dots, n$. For $i = 0$ we have $\Gamma_0(\sigma, \phi) = \emptyset = \Gamma_0(\sigma)$, and so the result is certainly true in this case. Now, for the inductive step, let $0 \leq i < n$ and suppose $\Gamma_i(\sigma, \phi) = \Gamma_i(\sigma)$. We must show $\Gamma_{i+1}(\sigma, \phi) = \Gamma_{i+1}(\sigma)$. We consider two cases.

Case (i): $\Gamma_i(\sigma) \cup \{\alpha_{n-i}\} \not\models \bot$.
Then $\Gamma_{i+1}(\sigma) = \Gamma_i(\sigma) \cup \{\alpha_{n-i}\} = \Gamma_i(\sigma, \phi) \cup \{\alpha_{n-i}\}$. Since $\Gamma_{i+1}(\sigma) \subseteq \Gamma_n(\sigma)$ and since $\phi \notin Cn(\Gamma_n(\sigma))$, we get also $\phi \notin Cn(\Gamma_{i+1}(\sigma))$, i.e., $\Gamma_i(\sigma, \phi) \cup \{\alpha_{n-i}\} \not\models \phi$. Hence $\Gamma_{i+1}(\sigma, \phi) = \Gamma_i(\sigma, \phi) \cup \{\alpha_{n-i}\} = \Gamma_{i+1}(\sigma)$ as required.
Case (ii): $\Gamma_i(\sigma) \cup \{\alpha_{n-i}\} \models \bot$.
Then $\Gamma_{i+1}(\sigma) = \Gamma_i(\sigma)$. Also, we get $\Gamma_i(\sigma) \cup \{\alpha_{n-i}\} \models \phi$. By the inductive hypothesis $\Gamma_i(\sigma, \phi) = \Gamma_i(\sigma)$, this is equivalent to $\Gamma_i(\sigma, \phi) \cup \{\alpha_{n-i}\} \models \phi$. Hence $\Gamma_{i+1}(\sigma, \phi) = \Gamma_i(\sigma, \phi) = \Gamma_i(\sigma)$. Thus $\Gamma_{i+1}(\sigma, \phi) = \Gamma_{i+1}(\sigma)$ as required. Hence we have shown $\Gamma_i(\sigma, \phi) = \Gamma_i(\sigma)$ for all $i = 0, 1, \ldots, n$. In particular this means $\Gamma_n(\sigma, \phi) = \Gamma_n(\sigma)$ and so $K \ominus_\sigma \phi = K$.
(L4) If $\models \phi_1 \leftrightarrow \phi_2$ then $K \ominus_\sigma \phi_1 = K \ominus_\sigma \phi_2$ Suppose $\models \phi_1 \leftrightarrow \phi_2$. Then either both $\models \phi_1$ and $\models \phi_2$ or both $\not\models \phi_1$ and $\not\models \phi_2$. In the former case we get $K \ominus_\sigma \phi_1 = K = K \ominus_\sigma \phi_2$ as required, while in the latter case we get $K \ominus_\sigma \phi_1 = Cn(\Gamma_n(\sigma, \phi_1))$ and $K \ominus_\sigma \phi_2 = Cn(\Gamma_n(\sigma, \phi_2))$. By an easy induction on i $\Gamma_i(\sigma, \phi_1) = \Gamma_i(\sigma, \phi_2)$ for all $i = 0, 1, \ldots, n$. In particular $\Gamma_n(\sigma, \phi_1) = \Gamma_n(\sigma, \phi_2)$.
(Failure) $K \ominus_\sigma \top = K$ Holds trivially. ■

We can also show that σ-liberation does satisfy a certain weaker form of (Inclusion), but for this we will wait until Sect. 4, after we have provided an axiomatic characterisation of σ-liberation.

3.2. Linear liberation

We now present a different way of using a sequence of sentences to define a retraction operator. These sequences are different from the σ used before, and will be employed in a simpler fashion. Intuitively, the agent has in mind several different candidate belief sets. We assume that the agent can order these candidate belief sets linearly according to preference, with the agent's actual current belief set identified with the most preferred belief set in this ordering. Since we work in a *finite* propositional language, every belief set can be identified with a single sentence. Therefore, we represent the agent's epistemic state as a sequence $\rho = (\beta_1, \ldots, \beta_m)$ of sentences, where each β_i stands for the belief set $Cn(\beta_i)$. $Cn(\beta_1)$ is the most preferred belief set, $Cn(\beta_2)$ is the next most preferred belief set, and so on[3].

DEFINITION 3.6. *Let K be a belief set and $\rho = (\beta_1, \ldots, \beta_m)$ a finite sequence of sentences. Then ρ is a K-*sequence *iff we have $K = Cn(\beta_1)$.*

[3]Since ρ is a *sequence* this means that the same sentence may appear more than once in ρ. However, for the results in this paper, this feature can be ignored if desired.

We used the natural numbers to index the sentences in the sequence ρ, but we could take them to be indexed by *any* totally ordered set. (This observation will turn out to be useful in the proof of Prop. 3.16.)

Now to remove a sentence ϕ from K using a K-sequence ρ we just take our new belief set to be the one generated by the most preferred sentence – according to ρ – which does not imply ϕ. If no such sentence exists, equivalently, if $\bigvee_k \beta_k \models \phi$, then we just take our new belief set to be K if ϕ is a tautology, and $Cn(\emptyset)$ otherwise. More precisely, from a given K-sequence ρ we define the operator $\eqsim_\rho$ for K by

$$K \eqsim_\rho \phi = \begin{cases} Cn(\beta_i) \text{ where } i = \min\{k \mid \beta_k \not\models \phi\} & \text{if } \bigvee_k \beta_k \not\models \phi \\ K & \text{if } \models \phi \\ Cn(\emptyset) & \text{otherwise} \end{cases}$$

DEFINITION 3.7. *Let K be a belief set and $\eqsim$ be an operator for K. Then $\eqsim$ is a* linear liberation *operator (for K) iff $\eqsim = \eqsim_\rho$ for some K-sequence ρ.*

K-sequences essentially correspond to the 'linear' variety of the type of general epistemic state considered by Alexander Bochman [2]. (Unlike us, Bochman also considers infinite languages.)

It turns out that linear liberation operators do not satisfy (Inclusion) either. For a simple counterexample let $K = Cn(p)$ and consider the K-sequence $\rho = (p, \neg p)$. Then $K \eqsim_\rho p = Cn(\neg p)$, so $\neg p$ has entered the belief set. The next proposition gives us some properties of linear liberation.

PROPOSITION 3.8. *Every linear liberation operator for K is a retraction operator for K which satisfies*

If $\theta \notin K \eqsim (\theta \wedge \phi)$ then $K \eqsim \theta = K \eqsim (\theta \wedge \phi)$ (Hyperregularity)

PROOF. Let $\eqsim$ be a linear liberation operator. By definition this means that $\eqsim = \eqsim_\rho$ for some K-sequence $\rho = (\beta_1, \ldots, \beta_m)$. To show $\eqsim_\rho$ is a retraction operator we check each of **(L1)**–**(L4)** and (Failure) in turn.

(L1) $K \eqsim_\rho \phi = Cn(K \eqsim_\rho \phi)$ By the definition of $K \eqsim_\rho \phi$ either $K \eqsim_\rho \phi = K$, in which case the rule holds since K is a belief set, or $K \eqsim_\rho \phi$ takes the form $Cn(\Gamma)$ for some $\Gamma \subseteq L$, in which case the rule holds by the idempotence of Cn.

(L2) If $\not\models \phi$ then $\phi \notin K \eqsim_\rho \phi$ This is obvious.

(L3) If $\phi \notin K$ then $K \eqsim_\rho \phi = K$ Suppose $\phi \notin K$. Then, since $K = Cn(\beta_1)$ by definition of a K-sequence, we have $\phi \notin Cn(\beta_1)$. Hence $\bigvee_k \beta_k \not\models \phi$ and so $K \eqsim_\rho \phi = Cn(\beta_i)$, where $i = \min\{k \mid \beta_k \not\models \phi\}$. But clearly $i = 1$, so $K \eqsim_\rho \phi = Cn(\beta_1) = K$ as required.

(L4) If $\models \phi_1 \leftrightarrow \phi_2$ then $K \ominus_\rho \phi_1 = K \ominus_\rho \phi_2$ If $\models \phi_1 \leftrightarrow \phi_2$ then we have $\Gamma \models \phi_1$ iff $\Gamma \models \phi_2$ for *any* set $\Gamma \subseteq L$. Hence, in the definition of $K \ominus_\rho \phi_1$ we can clearly replace ϕ_1 everywhere by ϕ_2, which means that $K \ominus_\rho \phi_1 = K \ominus_\rho \phi_2$ as required.
(Failure) $K \ominus_\rho \top = K$ Again holds trivially.
Hence we have shown $\ominus_\rho$ is a retraction operator. It remains to show $\ominus_\rho$ satisfies (Hyperregularity). So suppose $\theta \notin K \ominus_\rho (\theta \wedge \phi)$. Then we must have $\not\models \theta$ and so also $\not\models (\theta \wedge \phi)$. For the case $\bigvee_k \beta_k \models \theta \wedge \phi$ we must then have $K \ominus_\rho \theta = Cn(\emptyset) = K \ominus_\rho (\theta \wedge \phi)$ as required. So suppose $\bigvee_k \beta_k \not\models \theta \wedge \phi$. Then let $j = \min\{k \mid \beta_k \not\models \theta \wedge \phi\}$. We have $K \ominus_\rho (\theta \wedge \phi) = Cn(\beta_j)$. Furthermore, since $\theta \notin K \ominus_\rho (\theta \wedge \phi)$, we know $\beta_j \not\models \theta$. We claim also $j = \min\{k \mid \beta_k \not\models \theta\}$. This holds since if $j' < j$ and $\beta_{j'} \not\models \theta$ then obviously $\beta_{j'} \not\models \theta \wedge \phi$, but this contradicts $j = \min\{k \mid \beta_k \not\models \theta \wedge \phi\}$. Hence $j = \min\{k \mid \beta_k \not\models \theta\}$ as claimed, and so $K \ominus_\rho \theta = Cn(\beta_j) = K \ominus_\rho (\theta \wedge \phi)$ again as required. ∎

(Hyperregularity) comes from [10]. When added to the basic retraction postulates this rule allows us to derive some extra properties:

PROPOSITION 3.9. *Let $\ominus$ be a retraction operator which satisfies (Hyperregularity). Then $\ominus$ also satisfies the following two properties:*

- *Either $K \ominus (\theta \wedge \phi) = K \ominus \theta$ or $K \ominus (\theta \wedge \phi) = K \ominus \phi$*
- *If $\theta \notin K \ominus \phi$ and $\phi \notin K \ominus \theta$ then $K \ominus \theta = K \ominus \phi$*

PROOF. For the first property, suppose first that $\models \theta \wedge \phi$. Then, by **(L4)** and (Failure), $K \ominus (\theta \wedge \phi)$, $K \ominus \theta$ and $K \ominus \phi$ are all equal to K and so the property holds in this case. So assume instead $\not\models \theta \wedge \phi$. Now it cannot be the case that both θ **and** ϕ belong to $K \ominus (\theta \wedge \phi)$, since if so then we would have $\theta \wedge \phi \in K \ominus (\theta \wedge \phi)$ by **(L1)**, and this contradicts **(L2)**. Hence either $\theta \notin K \ominus (\theta \wedge \phi)$ or $\phi \notin K \ominus (\theta \wedge \phi)$. Applying (Hyperregularity) in the former case gives $K \ominus (\theta \wedge \phi) = K \ominus \theta$, while applying it in the latter case gives $K \ominus (\theta \wedge \phi) = K \ominus \phi$.[4] Thus the property holds also in this case.

For the second property, suppose $\theta \notin K \ominus \phi$ and $\phi \notin K \ominus \theta$. From the first property just proved above, we know either $K \ominus (\theta \wedge \phi) = K \ominus \theta$ or $K \ominus (\theta \wedge \phi) = K \ominus \phi$. Suppose the former holds. Then, from $\phi \notin K \ominus \theta$ we get $\phi \notin K \ominus (\theta \wedge \phi)$. Then, applying (Hyperregularity), we obtain $K \ominus \phi = K \ominus (\theta \wedge \phi) = K \ominus \theta$ as required. By a symmetric argument

[4] In our proofs we will not always mention explicitly the more obvious uses of **(L4)** such as $K \ominus (\theta \wedge \phi) = K \ominus (\phi \wedge \theta)$ or $K \ominus \phi = K \ominus \neg\neg\phi$.

we can show that the desired conclusion also obtains if we assume instead $K \ominus (\theta \wedge \phi) = K \ominus \phi$. ■

The first property above is the postulate known as (Decomposition) [1]. The second property gives a condition for when removing two different sentences yields the same result.

Prop. 3.8 gives us a sound list of postulates for linear liberation. We would now like to show that this list is complete, i.e., that *every* retraction operator $\ominus$ for K which satisfies (Hyperregularity) is of the form $\ominus_\rho$ for some K-sequence ρ. To do this we will describe how to construct, from a given such $\ominus$, a special K-sequence $\rho(\ominus)$. This construction will also be used in the next subsection when we come to characterising certain subclasses of the linear liberation operators. We define $\rho(\ominus) = (\beta_1, \ldots, \beta_m)$ from K and $\ominus$ inductively as follows:

(i) β_1 is chosen such that $Cn(\beta_1) = K$.

(ii) For $i > 0$, given we have defined $\beta_1, \ldots, \beta_i$, we choose β_{i+1} such that $Cn(\beta_{i+1}) = K \ominus (\bigvee_{j=1}^{i} \beta_j)$ (such a β_{i+1} exists since $K \ominus (\bigvee_{j=1}^{i} \beta_j)$ is deductively closed by **(L1)**).

(iii) m is minimal such that $\models \bigvee_{j=1}^{m} \beta_j$.

Notes: In step *(ii)* the precise choice of the β_j makes no difference by **(L4)**. Also, for each $i < m$, we have $\not\models \bigvee_{j=1}^{i} \beta_j$ by the minimality of m in step *(iii)*. Hence, by **(L2)** we have $\bigvee_{j=1}^{i} \beta_j \notin K \ominus (\bigvee_{j=1}^{i} \beta_j)$ and so $\beta_{i+1} \not\models \bigvee_{j=1}^{i} \beta_j$. Thus $\bigvee_{j=1}^{i+1} \beta_j \not\models \bigvee_{j=1}^{i} \beta_j$ and so the existence of m in step *(iii)* is guaranteed. Also note that if we follow the convention that an empty disjunction is logically equivalent to $\bot$ then we may write $Cn(\beta_1) = K = K \ominus \bot$[5] $= K \ominus (\bigvee_{j<1} \beta_j)$. Hence, for *all* $i = 1, \ldots, m$, $Cn(\beta_i) = K \ominus (\bigvee_{j<i} \beta_j)$. By step *(i)* the sequence $\rho(\ominus)$ is clearly a K-sequence. We then have the following:

Proposition 3.10. *Let $\ominus$ be a retraction operator for K which satisfies (Hyperregularity). Then $\ominus = \ominus_{\rho(\ominus)}$.*

Proof. Let $\rho(\ominus) = (\beta_1, \ldots, \beta_m)$. We need to show that, for all $\phi \in L$, $K \ominus_{\rho(\ominus)} \phi = K \ominus \phi$. Firstly, if $\models \phi$ then $K \ominus_{\rho(\ominus)} \phi = K$ by definition of $\ominus_{\rho(\ominus)}$, while $K \ominus \phi = K \ominus \top = K$ using **(L4)** and (Failure). Hence in this case we get the required conclusion. Now suppose $\not\models \phi$. Then, since $\models \bigvee_{j \leq m} \beta_j$, this means also $\bigvee_{j \leq m} \beta_j \not\models \phi$. Hence in this case $K \ominus_{\rho(\ominus)} \phi$ $= Cn(\beta_i)$, where i is minimal such that $\beta_i \not\models \phi$. Then, since $Cn(\beta_i) =$

[5] This follows from (Vacuity) and our assumption that K is consistent.

$K \ominus (\bigvee_{j<i} \beta_j)$, i is minimal such that $\phi \notin K \ominus (\bigvee_{j<i} \beta_j)$. By the minimality of i, $\beta_j \models \phi$ for all $j < i$, so $\models (\phi \wedge \bigvee_{j<i} \beta_j) \leftrightarrow (\bigvee_{j<i} \beta_j)$ and so, by **(L4)**, $\phi \notin K \ominus (\phi \wedge \bigvee_{j<i} \beta_j)$. Applying (Hyperregularity) to this gives us $K \ominus \phi = K \ominus (\phi \wedge \bigvee_{j<i} \beta_j)$, and so, re-applying **(L4)** to the right-hand side, $K \ominus \phi = K \ominus (\bigvee_{j<i} \beta_j) = Cn(\beta_i) = K \ominus_{\rho(\ominus)} \phi$ as required. ∎

Propositions 3.8 and 3.10 together give us the following characterisation for linear liberation operators (cf. Representation Theorem 5 in [2]).

THEOREM 3.11. *For a given belief set K, $\ominus$ is a linear liberation operator iff $\ominus$ is a retraction operator which satisfies (Hyperregularity).*

3.3. Special cases of linear liberation

Note that, in the definition of a K-sequence, there need not be *any* relationship between the sentences β_i. Other, more restricted classes of liberation operators can now be found by placing restrictions on the β_i. We consider four here. First it is natural to ask: when does a linear liberation operator $\ominus_\rho$ satisfy (Inclusion)? It is quite easy to see that this will happen if and only if each sentence in ρ is a logical consequence of β_1, i.e.,

(A) For each $i = 1, \ldots, m$, $\beta_1 \models \beta_i$

PROPOSITION 3.12. *Let $\ominus$ be a linear liberation operator for K. Then $\ominus$ satisfies (Inclusion) iff $\ominus = \ominus_\rho$ for some K-sequence ρ satisfying (A).*

PROOF. To show the 'if' direction, first note that (Inclusion) always holds if $\bigvee_k \beta_k \models \phi$ (since in this case $K \ominus_\rho \phi$ is equal to either K or $Cn(\emptyset)$), while if $\bigvee_k \beta_k \not\models \phi$ we have $K \ominus_\rho \phi = Cn(\beta_i)$ for some i. Then, since $\beta_1 \models \beta_i$ by (A), we have $Cn(\beta_i) \subseteq Cn(\beta_1)$ and so $K \ominus_\rho \phi \subseteq Cn(\beta_1) = K$.

For the 'only if' direction suppose $\ominus$ satisfies (Inclusion). By Prop. 3.10 $\ominus = \ominus_{\rho(\ominus)}$ where $\rho(\ominus)$ is the K-sequence constructed earlier. We will simply show that if $\ominus$ satisfies (Inclusion) then $\rho(\ominus)$ satisfies (A). But, for each $i = 1, \ldots, m$, the construction gave us $Cn(\beta_i) = K \ominus (\bigvee_{j<i} \beta_j)$. Hence, using (Inclusion), we deduce $Cn(\beta_i) \subseteq K = Cn(\beta_1)$. Hence $\beta_1 \models \beta_i$ as required. ∎

Next consider the following, stronger, condition on a K-sequence $\rho = (\beta_1, \ldots, \beta_m)$:

(B) For $i < j$ we have $\beta_i \models \beta_j$

(B) – which says that sentences get progressively logically weaker through ρ – leads to an important class of withdrawal operators – the class of *severe withdrawal* operators which, as is shown in [17], may be characterised by the basic retraction postulates plus (Inclusion) and the following two rules:[6]

- If $\not\models \theta$ then $K \ominus \theta \subseteq K \ominus (\theta \wedge \phi)$ (Antitony)
- If $\theta \notin K \ominus (\theta \wedge \phi)$ then $K \ominus (\theta \wedge \phi) \subseteq K \ominus \theta$ (Conjunctive Inclusion)

Note that the second rule above corresponds to 'one half' of (Hyperregularity) and is an AGM *supplementary postulate* for contraction [1]. The first rule above is a strengthened version of the other supplementary postulate "$(K \ominus \theta) \cap (K \ominus \phi) \subseteq K \ominus (\theta \wedge \phi)$".

PROPOSITION 3.13. *Let K be a belief set and $\ominus$ an operator for K. Then $\ominus = \ominus_\rho$ for some K-sequence ρ which satisfies (B) iff $\ominus$ is a severe withdrawal operator.*

PROOF. For the 'only if' direction let ρ be a K-sequence which satisfies (B). By Prop. 3.8 we know already that $\ominus_\rho$ is a retraction operator which satisfies (Hyperregularity). Since (Hyperregularity) implies (Conjunctive Inclusion) we know that the latter property is satisfied. Since (B) implies (A) we know (Inclusion) is satisfied by Prop. 3.12. It remains to show that (Antitony) is satisfied. So suppose $\not\models \theta$. If $\bigvee_k \beta_k \models \theta$ then $K \ominus_\rho \theta = Cn(\emptyset)$ so we get $K \ominus_\rho \theta \subseteq K \ominus_\rho (\theta \wedge \phi)$ as required. So suppose $\bigvee_k \beta_k \not\models \theta$ (so also $\bigvee_k \beta_k \not\models \theta \wedge \phi$). Let $j_1 = \min\{k \mid \beta_k \not\models \theta\}$ and $j_2 = \min\{k \mid \beta_k \not\models \theta \wedge \phi\}$. Since $\beta_{j_1} \not\models \theta$ we also know $\beta_{j_1} \not\models \theta \wedge \phi$ and so, by the minimality of j_2, we must have $j_2 \leq j_1$. Hence (B) tells us $\beta_{j_2} \models \beta_{j_1}$ and so $K \ominus_\rho \theta = Cn(\beta_{j_1}) \subseteq Cn(\beta_{j_2}) = K \ominus_\rho (\theta \wedge \phi)$, again as required. Hence $\ominus_\rho$ is a severe withdrawal operator.

For the 'if' direction let $\ominus$ be a severe withdrawal operator. Since (Antitony) and (Conjunctive Inclusion) jointly imply (Hyperregularity), $\ominus$ is also a retraction liberation operator which satisfies (Hyperregularity). Hence, by Prop. 3.10, $\ominus = \ominus_{\rho(\ominus)}$. We will show that the fact that $\ominus$ satisfies (Inclusion) and (Antitony) is enough to ensure that $\rho(\ominus)$ satisfies (B). So let $i < j$. We must show $\beta_i \models \beta_j$. If $i = 1$ then, since $\ominus$ satisfies (Inclusion), we know $\beta_1 \models \beta_j$ from the proof of Prop. 3.12. So suppose $1 < i < j$. Then, from the construction of $\rho(\ominus)$, $Cn(\beta_j) = K \ominus \bigvee_{k<j} \beta_k$ and $Cn(\beta_i) = K \ominus \bigvee_{k<i} \beta_k$. Since $i < j$, $\models (\bigvee_{k<i} \beta_k) \leftrightarrow (\bigvee_{k<i} \beta_k \wedge \bigvee_{k<j} \beta_k)$ and hence, by **(L4)**, $Cn(\beta_i) = K \ominus (\bigvee_{k<i} \beta_k \wedge \bigvee_{k<j} \beta_k)$. Using (Antitony) then gives us $Cn(\beta_j) \subseteq Cn(\beta_i)$ and so $\beta_i \models \beta_j$ as required. ■

[6]Such sequences are also studied in [6] from the perspective of qualitative utility in economics.

An example of a condition that doesn't lead to the satisfaction of (Inclusion) is the following:

(C) $\models \bigvee_k \beta_k$ and, for $i \neq j$, $\beta_i \wedge \beta_j$ is inconsistent

(C) says that the sentences in ρ represent mutually incompatible points of view. Moreover, the different points of view have nothing but tautologies in common (since $\models \bigvee_k \beta_k$, or equivalently, $\bigcap_k Cn(\beta_k) = Cn(\emptyset)$).

Proposition 3.14. *Let K be a belief set and $\Bumpeq$ an operator for K. Then $\Bumpeq = \Bumpeq_\rho$ for some K-sequence ρ which satisfies (C) iff $\Bumpeq$ is a linear liberation operator that satisfies*

If $(K \Bumpeq \theta) \cup (K \Bumpeq \phi)$ is consistent then $K \Bumpeq \theta = K \Bumpeq \phi$ (Dichotomy)

Proof. For the 'only if' direction let ρ be a K-sequence which satisfies (C). First note that $\models \bigvee_k \beta_k$ implies for all $\psi \in L$, either $K \Bumpeq_\rho \psi = Cn(\beta_i)$ where i is minimal such that $\beta_i \not\models \psi$, or (if $\models \psi$) $K \Bumpeq_\rho \psi = K$. Since $K = Cn(\beta_1)$, $K \Bumpeq_\rho \psi$ *always* takes the form $Cn(\beta_i)$ for some $i = 1, \ldots, m$. So let i, j be such that $K \Bumpeq_\rho \theta = Cn(\beta_i)$ and $K \Bumpeq_\rho \phi = Cn(\beta_j)$. If $i = j$ then $K \Bumpeq_\rho \theta = K \Bumpeq_\rho \phi$ and so (Dichotomy) holds. If $i \neq j$ then, by (C), $\beta_i \wedge \beta_j$ is inconsistent and so $Cn(\beta_i) \cup Cn(\beta_j)$ is inconsistent, i.e., $(K \Bumpeq_\rho \theta) \cup (K \Bumpeq_\rho \phi)$ is inconsistent. Thus also in this case (Dichotomy) holds.

For the 'if' direction, let $\Bumpeq$ be a linear liberation operator which satisfies (Dichotomy). We will show that the fact that $\Bumpeq$ satisfies (Dichotomy) is enough to ensure that the K-sequence $\rho(\Bumpeq) = (\beta_1, \ldots, \beta_m)$ satisfies (C). Since $\models \bigvee_k \beta_k$, it remains to show that, for $i \neq j$, $\beta_i \wedge \beta_j$ is inconsistent. For each $i = 1, \ldots, m$ $Cn(\beta_i) = K \Bumpeq \bigvee_{k<i} \beta_k$. Suppose $i \neq j$. Assume, without loss of generality, that $i < j$. Then $Cn(\beta_i) = K \Bumpeq \bigvee_{k<i} \beta_k$ and $Cn(\beta_j) = K \Bumpeq \bigvee_{k<j} \beta_k$. Since $i < j$, $\bigvee_{k<j} \beta_k \in Cn(\beta_i)$, i.e., $\bigvee_{k<j} \beta_k \in K \Bumpeq \bigvee_{k<i} \beta_k$. Since $\bigvee_{k<j} \beta_k \notin K \Bumpeq \bigvee_{k<j} \beta_k$, $K \Bumpeq \bigvee_{k<i} \beta_k \neq K \Bumpeq \bigvee_{k<j} \beta_k$. Hence, by (Dichotomy), $(K \Bumpeq \bigvee_{k<i} \beta_k) \cup (K \Bumpeq \bigvee_{k<j} \beta_k)$ is inconsistent, i.e., $Cn(\beta_i) \cup Cn(\beta_j)$ is inconsistent. Since $Cn(Cn(\beta_i) \cup Cn(\beta_j)) = Cn(\beta_i \wedge \beta_j)$ this is equivalent to saying $\beta_i \wedge \beta_j$ is inconsistent as required. ■

A justification for (Dichotomy) is provided by the condition on the sequence ρ. An agent has a sequence of mutually incompatible belief sets. Its way of dealing with changes will necessarily have to be dichotomous. When is such a mode of reasoning sensible? When the agent has become quite sophisticated through a process of refinement and ironing out differences in its belief corpus. The theories in ρ, then, are most plausibly viewed as the end products of a period of making small changes and converging on a cluster of (incompatible) alternatives. Therefore, we refer to this type of liberation as *dichotomous* liberation. (Dichotomy) can also be seen as describing belief change that lies between contraction and revision – a view confirmed by the discussion in Sect. 5.1.

Finally we have the following condition:

(D) For $i < j$ either $\beta_i \models \beta_j$ or $\beta_i \wedge \beta_j \models \bigvee_{k<i} \beta_k$

Each of (B) and (C) implies (D), a condition which leads us to the following subclass of linear liberation:

PROPOSITION 3.15. *Let K be a belief set and $\ominus$ an operator for K. Then $\ominus = \ominus_\rho$ for some K-sequence ρ which satisfies (D) iff $\ominus$ is a linear liberation operator that satisfies*

$$\textit{If } (K \ominus \theta) \cup (K \ominus \phi) \not\models \phi \textit{ then } K \ominus \theta \subseteq K \ominus \phi \qquad \textit{(Strong Conservativity)}$$

PROOF. For the 'only if' direction let $\rho = (\beta_1, \ldots, \beta_m)$ be a K-sequence which satisfies (D) and suppose $(K \ominus_\rho \theta) \cup (K \ominus_\rho \phi) \not\models \phi$. Then clearly $\not\models \phi$. We now consider two cases:
Case *(i)*: $\bigvee_k \beta_k \models \phi$ In this case $K \ominus_\rho \phi = Cn(\emptyset)$ and so $(K \ominus_\rho \theta) \cup (K \ominus_\rho \phi) = (K \ominus_\rho \theta) \cup Cn(\emptyset) = K \ominus_\rho \theta$. Hence the assumption that $(K \ominus_\rho \theta) \cup (K \ominus_\rho \phi) \not\models \phi$ reduces to $K \ominus_\rho \theta \not\models \phi$. We now claim $K \ominus_\rho \theta = Cn(\emptyset)$. To see this, note that if either $\bigvee_k \beta_k \not\models \theta$ or $\models \theta$ then $K \ominus_\rho \theta = Cn(\beta_i)$ for some i (remembering that $K = Cn(\beta_1)$ for the case $\models \theta$) and so, from $K \ominus_\rho \theta \not\models \phi$, we get $\phi \notin Cn(\beta_i)$ which contradicts our assumption that $\bigvee_k \beta_k \models \phi$. Hence both $\bigvee_k \beta_k \models \theta$ and $\not\models \theta$. By definition of $\ominus_\rho$, then, we have $K \ominus_\rho \theta = Cn(\emptyset)$ as claimed. Hence we obtain $K \ominus_\rho \theta = K \ominus_\rho \phi$.
Case *(ii)*: $\bigvee_k \beta_k \not\models \phi$ In this case $K \ominus_\rho \phi = Cn(\beta_i)$ where i is minimal such that $\beta_i \not\models \phi$. Now if both $\not\models \theta$ and $\bigvee_k \beta_k \models \theta$ then $K \ominus_\rho \theta = Cn(\emptyset) \subseteq K \ominus_\rho \phi$ and so we get the required conclusion. So suppose either $\models \theta$ or $\bigvee_k \beta_k \not\models \theta$. Then $K \ominus_\rho \theta = Cn(\beta_j)$ for some j. Now if this j were such that $j < i$ then, by minimality of i, we would have $\beta_j \models \phi$ and so $\phi \in K \ominus_\rho \theta$. But this implies $(K \ominus_\rho \theta) \cup (K \ominus_\rho \phi) \models \phi$, contradicting our initial assumption. Hence $i \leq j$. In case $i = j$ we get $K \ominus_\rho \theta = K \ominus_\rho \phi$ which gives the required conclusion. Suppose now that $i < j$. Then, by (D), either $\beta_i \models \beta_j$ or $\beta_i \wedge \beta_j \models \bigvee_{k<i} \beta_k$. But if the latter were true we would deduce $\beta_i \wedge \beta_j \models \phi$ (since $\bigvee_{k<i} \beta_k \models \phi$ using the minimality of i) and so $Cn(\beta_i) \cup Cn(\beta_j) \models \phi$, i.e., $(K \ominus_\rho \theta) \cup (K \ominus_\rho \phi) \models \phi$. This again contradicts the initial assumption.

Hence $\beta_i \models \beta_j$, and so, since this is equivalent to $Cn(\beta_j) \subseteq Cn(\beta_i)$, we again get our required conclusion.

For the ‘if’ direction let $\Bumpeq$ be a linear liberation operator satisfying (Strong Conservativity). We show that $\rho(\Bumpeq) = (\beta_1, \dots, \beta_m)$ satisfies (D). Let $i < j$. Then $Cn(\beta_i) = K \Bumpeq \bigvee_{k<i} \beta_k$ and $Cn(\beta_j) = K \Bumpeq \bigvee_{k<j} \beta_k$. By (Strong Conservativity) we know either $(K \Bumpeq \bigvee_{k<j} \beta_k) \cup (K \Bumpeq \bigvee_{k<i} \beta_k) \models \bigvee_{k<i} \beta_k$ or $K \Bumpeq \bigvee_{k<j} \beta_k \subseteq K \Bumpeq \bigvee_{k<i} \beta_k$. If the former holds then we get $Cn(\beta_j) \cup Cn(\beta_i) \models \bigvee_{k<i} \beta_k$ which gives $\beta_i \wedge \beta_j \models \bigvee_{k<i} \beta_k$ and so (D) holds. If the latter holds then we get $Cn(\beta_j) \subseteq Cn(\beta_i)$ which gives $\beta_i \models \beta_j$ and so again (D) holds. ■

The postulate (Strong Conservativity) has been shown to be a characteristic postulate for *base-generated maxichoice contraction* operators [10]. The significance of this particular subclass of linear liberation operators is that it is equivalent to none other than the class of σ-liberation operators from Sect. 3.1:

THEOREM 3.16. *Let K be a belief set. Then for each belief sequence σ relative to K there exists a K-sequence ρ satisfying (D) such that $\Bumpeq_\sigma = \Bumpeq_\rho$. Conversely for each K-sequence ρ satisfying (D) there exists a belief sequence σ relative to K such that $\Bumpeq_\rho = \Bumpeq_\sigma$.*

PROOF. From σ to ρ. Let $\sigma = (\alpha_1, \dots, \alpha_n)$ be a given belief sequence relative to K. We will construct from σ a suitable K-sequence ρ. First recall that the sentences in ρ may be indexed by any totally ordered set. We take a special set of indices, namely the set $Con(\llbracket\sigma\rrbracket)$ of all *consistent* subsets of the set $\llbracket\sigma\rrbracket$ of all sentences appearing in σ. Furthermore, we take this set to be ordered by the relation $\prec$ defined by setting, for all $X, Y \in Con(\llbracket\sigma\rrbracket)$,

$$X \prec Y \quad \text{iff} \quad \text{there is some } i \text{ such that } \alpha_i \in X \setminus Y \text{ and, for all } j > i,\ \alpha_j \in X \text{ iff } \alpha_j \in Y$$

$\prec$ forms a strict total order on $Con(\llbracket\sigma\rrbracket)$ and satisfies the following property, for all $X, Y \in Con(\llbracket\sigma\rrbracket)$:

$$\text{If } X \supset Y \text{ then } X \prec Y. \tag{i}$$

Given this index set $Con(\llbracket\sigma\rrbracket)$ we then specify a sequence $\rho = (\beta_X)_{X \in Con(\llbracket\sigma\rrbracket)}$ by setting, for each $X \in Con(\llbracket\sigma\rrbracket)$, $\beta_X = \bigwedge X$. (The precise ordering in which the sentences in X appear in this conjunction does not matter.) Note that since $\emptyset \in Con(\llbracket\sigma\rrbracket)$ and $\models \beta_\emptyset$ (following the convention that an empty conjunction is logically equivalent to $\top$), we have $\models \bigvee_{X \in Con(\llbracket\sigma\rrbracket)} \beta_X$. We now need to show three things: *(a)* The sequence ρ satisfies (D), *(b)* ρ is a K-sequence, i.e., $K = Cn(\beta_{X_0})$, where X_0 is minimal under $\prec$ in $Con(\llbracket\sigma\rrbracket)$, and *(c)* $K \circledast_\rho \phi = K \circledast_\sigma \phi$ for all $\phi \in L$.

To show ρ satisfies (D), we need to show that, for all $X, Y \in Con(\llbracket\sigma\rrbracket)$,

$$\text{If } X \prec Y \text{ then either } \beta_X \models \beta_Y \text{ or } \beta_X \wedge \beta_Y \models \bigvee_{\{Z \in Con(\llbracket\sigma\rrbracket) \mid Z \prec X\}} \beta_Z.$$

So suppose $X, Y \in Con(\llbracket\sigma\rrbracket)$ and $X \prec Y$. Firstly, if $Y \subseteq X$ then $\bigwedge X \models \bigwedge Y$, i.e., $\beta_X \models \beta_Y$ as required. So assume instead $Y \not\subseteq X$. Now if $X \cup Y$ is inconsistent then so is $\bigwedge X \wedge \bigwedge Y = \beta_X \wedge \beta_Y$. Hence in this case we get $\beta_X \wedge \beta_Y \models \bigvee_{\{Z \in Con(\llbracket\sigma\rrbracket) \mid Z \prec X\}} \beta_Z$ as required. Now suppose $X \cup Y$ is consistent. Then, since $Y \not\subseteq X$, we have $X \subset X \cup Y$ and so, by (i) above, we know $X \cup Y \prec X$. Clearly $\models (\beta_X \wedge \beta_Y) \leftrightarrow \beta_{X \cup Y}$. Hence in this case (putting $Z' = X \cup Y$) $\models (\beta_X \wedge \beta_Y) \leftrightarrow \beta_{Z'}$ for some $Z' \in Con(\llbracket\sigma\rrbracket)$ such that $Z' \prec X$, which suffices to show $\beta_X \wedge \beta_Y \models \bigvee_{\{Z \in Con(\llbracket\sigma\rrbracket) \mid Z \prec X\}} \beta_Z$.

Thus *(a)* holds. To show *(b)* and *(c)*, we require the following:

LEMMA 3.17. *Let $\phi \in L$ be such that $\not\models \phi$. Then $\Gamma_n(\sigma, \phi) = X$, where X is minimal under $\prec$ in $Con(\llbracket\sigma\rrbracket)$ such that $X \not\models \phi$.*

PROOF. Since $\Gamma_n(\sigma, \phi) \not\models \phi$, it remains to show that $Y \prec \Gamma_n(\sigma, \phi)$ implies $Y \models \phi$ for all $Y \in Con(\llbracket\sigma\rrbracket)$. But if $Y \prec \Gamma_n(\sigma, \phi)$ then there exists i such that $\alpha_i \in Y \setminus \Gamma_n(\sigma, \phi)$ and, for all $j > i$ $\alpha_j \in Y$ iff $\alpha_j \in \Gamma_n(\sigma, \phi)$. By construction of $\Gamma_n(\sigma, \phi)$, since $\alpha_i \notin \Gamma_n(\sigma, \phi)$, $\Gamma_{n-i}(\sigma, \phi) \cup \{\alpha_i\} \models \phi$. Since Y contains α_i along with all elements of $\Gamma_n(\sigma, \phi)$ of the form α_j for $j > i$, $\Gamma_{n-i}(\sigma, \phi) \cup \{\alpha_i\} \subseteq Y$ and so $Y \models \phi$ as required. ■

Given this lemma we can now confirm that ρ is a K-sequence. Let X_0 be the minimal element under $\prec$ in $Con(\llbracket\sigma\rrbracket)$. Since obviously $X_0 \not\models \bot$, X_0 is also minimal under $\prec$ in $Con(\llbracket\sigma\rrbracket)$ such that $X_0 \not\models \bot$. Applying Lemma 3.17 then gives us $X_0 = \Gamma_n(\sigma, \bot) = \Gamma_n(\sigma)$. Hence $Cn(X_0) = Cn(\Gamma_n(\sigma)) = K$ as required. Thus *(b)* holds. It only remains to show *(c)* $K \circledast_\rho \phi = K \circledast_\sigma \phi$ for all $\phi \in L$. If $\models \phi$ then $K \circledast_\rho \phi = K = K \circledast_\sigma \phi$. If $\not\models \phi$ then, since $\models \bigvee_X \beta_X$, $\bigvee_X \beta_X \not\models \phi$. Thus $K \circledast_\rho \phi = Cn(\beta_X)$, where X is minimal in $Con(\llbracket\sigma\rrbracket)$ under $\prec$ such that $\beta_X \not\models \phi$, equivalently $X \not\models \phi$. But by Lemma 3.17 $X = \Gamma_n(\sigma, \phi)$. Hence $K \circledast_\rho \phi = Cn(\Gamma_n(\sigma, \phi)) = K \circledast_\sigma \phi$ as required.

<u>From ρ to σ.</u> Let $\rho = (\beta_1, \ldots, \beta_m)$ be a given K-sequence which satisfies (D). Then we define the belief sequence σ from ρ by simply reversing the sequence, i.e., we set $\sigma = (\beta_m, \ldots, \beta_1)$.

LEMMA 3.18. *Let $\phi \in L$ be such that $\bigvee_k \beta_k \not\models \phi$. Then $Cn(\Gamma_m(\sigma, \phi)) = Cn(\beta_i)$, where $i = \min\{k \mid \beta_k \not\models \phi\}$.*

PROOF. Let $i = \min\{k \mid \beta_k \not\models \phi\}$. Following the construction of $\Gamma_m(\sigma, \phi)$ we see that $\Gamma_k(\sigma, \phi) = \emptyset$ for all $k < i$ and $\Gamma_i(\sigma, \phi) = \{\beta_i\}$. Since ρ satisfies (D), for all $j > i$ either $\beta_i \models \beta_j$ or $\beta_i \wedge \beta_j \models \bigvee_{k<i} \beta_k$. By the minimality of i, $\bigvee_{k<i} \beta_k \models \phi$, hence this latter implies $\{\beta_i\} \cup \{\beta_j\} \models \phi$. Thus in the construction of $\Gamma_m(\sigma, \phi)$ from stage i, the only sentences added are those logically implied by β_i. Thus $Cn(\Gamma_m(\sigma, \phi)) = Cn(\beta_i)$ as required. ■

From this result we can see that σ is a belief sequence relative to K, for $Cn(\Gamma_m(\sigma)) = Cn(\Gamma_m(\sigma, \bot)) = Cn(\beta_i)$, where i is minimal such that β_i is consistent. Since $K = Cn(\beta_1)$ is consistent, $i = 1$ and so $Cn(\Gamma_m(\sigma)) = K$. It remains to show $K \ominus_\sigma \phi = K \ominus_\rho \phi$ for all $\phi \in L$. If $\models \phi$ then $K \ominus_\sigma \phi = K = K \ominus_\rho \phi$. If $\not\models \phi$ but $\bigvee_k \beta_k \models \phi$ then $K \ominus_\rho \phi = Cn(\emptyset)$ while, since $\Gamma_m(\sigma, \phi) = \emptyset$, also $K \ominus_\sigma \phi = Cn(\Gamma_m(\sigma, \phi)) = Cn(\emptyset)$ as required. Finally if $\bigvee_k \beta_k \not\models \phi$ then the conclusion follows directly from Lemma 3.18. ■

Given Theorem 3.16 we may state:

COROLLARY 3.19. *Let K be a belief set and let $\ominus$ be an operator for K. Then $\ominus$ is a σ-liberation operator iff $\ominus$ is a linear liberation operator that satisfies (Strong Conservativity).*

So σ-liberation may be axiomatically characterised by the basic retraction postulates plus (Hyperregularity) and (Strong Conservativity). Furthermore the results of this section allow us to say more: Every severe withdrawal operator is also a σ-liberation operator (as is every dichotomous liberation operator). Secondly, since severe withdrawal doesn't satisfy (Recovery), σ-liberation doesn't satisfy (Recovery) either.

4. Weaker versions of (Inclusion)

While we reject (Inclusion), we intend to do justice to the Principle of Minimal Change. Thus it behooves us to look for weaker versions which disallow gratuitous addition of new beliefs. In this section we consider some potential weakenings. At the end of this section we will check whether our proposed liberation operators satisfy these weakenings. Throughout this section, unless stated otherwise, $\ominus$ is assumed to be a retraction operator for K. The first weakening is the following:

(w1) If $\theta \in K \ominus \phi$ and $\theta \notin K$ then $\neg\theta \in K$

This rule states that a new sentence θ may be introduced into a belief set during a removal operation *only if* its negation was present before the removal (and, since $K \ominus \phi$ is always consistent,[7] has necessarily been given up during the removal). This formalises the intuition that a new sentence is introduced only if there was previously something present in the belief set which had kept it out but which is now no longer there.

Though **(w1)** looks reasonable at first glance, the following indicates it is *too* strong for our purposes.

PROPOSITION 4.1. *If (Closure) holds then* **(w1)** *is equivalent to:*

(w1′) *If K is not complete then $K \ominus \phi \subseteq K$*

PROOF. To show **(w1)** implies **(w1′)** suppose K is not complete[8]. Then there exists some $\lambda \in L$ such that $\lambda \notin K$ and $\neg\lambda \notin K$. If there existed $\theta \in (K \ominus \phi) \setminus K$ then $\theta \notin K$ would give us either $\theta \vee \lambda \notin K$ or $\theta \vee \neg\lambda \notin K$ (since K is deductively closed). Suppose the former. Then, since $\theta \vee \lambda \in K \ominus \phi$ (which follows from $\theta \in K \ominus \phi$ and (Closure)), we apply **(w1)** to obtain $\neg(\theta \vee \lambda) \in K$ and so $\neg\lambda \in K$. In a similar way if we suppose $\theta \vee \neg\lambda \in K \ominus \phi$ then we obtain $\lambda \in K$. Either way we get a contradiction and so there is no $\theta \in (K \ominus \phi) \setminus K$, i.e., $K \ominus \phi \subseteq K$ as required. To show **(w1′)** implies **(w1)** suppose $\theta \in (K \ominus \phi) \setminus K$. Then $K \ominus \phi \not\subseteq K$ and so, applying **(w1′)**, K is complete. Hence, from $\theta \notin K$ we get $\neg\theta \in K$ as required. ■

The rule **(w1′)** says that (Inclusion) holds whenever the prior belief set K is not complete. Since the prior belief set K typically will *not* be complete, **(w1′)** isn't much of a weakening of (Inclusion) and we should not be too disappointed when a suggested operation of retraction does not satisfy it (or the equivalent **(w1)**). A relaxed version of **(w1)** is:

(w2) If $\theta \in (K \ominus \phi) \setminus K$ then there exists $\psi \in K \ominus \phi$ such that $\psi \models \theta$ and $\neg\psi \in K$.

That is, every $\theta \in (K \ominus \phi) \setminus K$ can be 'traced back' to, i.e., is a *logical consequence* of, a sentence whose negation was in K but which is now included in $K \ominus \theta$. Equivalently:

PROPOSITION 4.2. *If (Closure) holds then* **(w2)** *is equivalent to:*

(w2′) *If $(K \ominus \phi) \cup K$ is consistent then $K \ominus \phi \subseteq K$*

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[7] This is already ensured by the (Success) postulate.

[8] A belief set K is complete iff for all $\lambda \in L$ either $\lambda \in K$ or $\neg\lambda \in K$.

Proof. To show **(w2)** implies **(w2′)** suppose $K \Bumpeq \phi \nsubseteq K$. Then there exists $\theta \in (K \Bumpeq \phi) \setminus K$ and so, by **(w2)**, there exists $\psi \in K \Bumpeq \phi$ such that $\neg\psi \in K$. Hence $(K \Bumpeq \phi) \cup K$ is inconsistent as required. To show **(w2′)** implies **(w2)** suppose there exists $\theta \in (K \Bumpeq \phi) \setminus K$. Then $K \Bumpeq \phi \nsubseteq K$ and so, applying **(w2′)**, $(K \Bumpeq \phi) \cup K$ is inconsistent. So, there is a sentence λ such that $\neg\lambda \in K$ and $\lambda \in K \Bumpeq \phi$.[9] Then consider the sentence $\lambda \wedge \theta$. We have $\lambda \wedge \theta \models \theta$, $\neg(\lambda \wedge \theta) \in K$ (since $\neg\lambda \in K$ and K is deductively closed) and $\lambda \wedge \theta \in K \Bumpeq \phi$ (since $\lambda, \theta \in K \Bumpeq \phi$ and $K \Bumpeq \phi$ is deductively closed by (Closure)). This suffices to prove **(w2)** holds. ■

(w2′) says: the new belief set is either included in the old one, or the agent now believes the negation of a sentence it previously held to be true. That is, if the agent does not weaken its belief set, it has made a complete about-turn regarding some beliefs.

Our last weakening of (Inclusion) is a property often held to be characteristic of withdrawal operators. When one removes a sentence θ from K using an operation $\Bumpeq$ of withdrawal, one does so without insisting that its negation $\neg\theta$ be in the new belief set. There is just one possible situation when $\neg\theta \in K \Bumpeq \theta$, and that is if $\neg\theta \in K$ (in which case – assuming as we do that K is consistent – $\theta \notin K$ and so $K \Bumpeq \theta = K$ by (Vacuity)). That is, the following rule is taken to hold:

(w3) If $\neg\theta \notin K$ then $\neg\theta \notin K \Bumpeq \theta$

How do σ-liberation and linear liberation fare with respect to the above weak (Inclusion) rules? Example 3.4 shows that σ-liberation (and hence also linear liberation) does not satisfy the weaker version **(w3)** since we have $\neg p \notin K$ but $\neg p \in K \Bumpeq_\sigma p$. Hence σ-liberation can result in the addition of the negation of the sentence being removed. This example also shows that σ-liberation doesn't satisfy **(w1)** (or, therefore, **(w1′)**), since we have $(\neg p \wedge \neg q) \vee r \in (K \Bumpeq_\sigma p) \setminus K$ but $\neg((\neg p \wedge \neg q) \vee r) \notin K$ for any propositional variable r distinct from p, q. However, note that the weak inclusion postulate **(w2′)** is just a special instance of (Strong Conservativity) (remembering that $K \Bumpeq \bot = K$ for any retraction operator for K). Hence we can see that σ-liberation operators *do* satisfy **(w2′)** (and the equivalent **(w2)**). However, linear liberation operators do not satisfy this property in general, as can be seen by taking $K = Cn(p)$ with the K-sequence $\rho = (p, q)$. Then, since $K \Bumpeq_\rho p = Cn(q)$ we have $(K \Bumpeq_\rho p) \cup K$ is consistent but $K \Bumpeq_\rho p \nsubseteq K$.

[9] E.g. let $\neg\lambda$ be the sentence characterised by $M(K)$. λ defined as such always exists, since we assume a finitely generated language.

5. From retraction to withdrawal and revision

In this section we consider the relationship between retraction operators and the traditional belief change operators of withdrawal and revision. In particular we show how retraction operators can be 'converted' into either withdrawal or revision operators.

5.1. Retraction to withdrawal

What distinguishes retraction operators from withdrawal operators is that removing beliefs using the former may lead to the introduction of *new* beliefs into the belief set, while using the latter *always* leads to a new belief set which is a subset of the prior belief set. However, there is a simple way in which a given retraction operator may be *transformed* into a withdrawal operator. After retraction is performed, we simply discard all sentences which were not originally elements of K, i.e., from each retraction operator $\Bumpeq$ for K we can define the new operator $\bumpeq$ for K by setting for each $\phi \in L$,

$$K \bumpeq \phi = K \cap (K \Bumpeq \phi).$$

Obviously $\bumpeq$ is guaranteed to satisfy (Inclusion). This is strongly reminiscent of the *Harper Identity* [8]:

(Harper Identity) $K \bumpeq \phi = K \cap (K * \neg\phi)$

where $*$ is a given *revision* operator. A formal difference is the appearance of 'ϕ' rather than '$\neg\phi$' on the right-hand side. A more crucial difference is that while the Harper Identity is usually employed as a means of obtaining a withdrawal operation from a given revision operator, here we use a slight variant of it to obtain a withdrawal operator from a *retraction* operator. Continuing with our liberation metaphor, we make the following definition:

Definition 5.1. *Let K be a belief set and let $\Bumpeq$ be an operator for K. If the operator $\bumpeq$ for K is defined from $\Bumpeq$ as above then we call $\bumpeq$ the* incarceration[10] *of $\Bumpeq$.*

As well as (Inclusion), the incarceration of a retraction operator satisfies (Closure), (Success), (Vacuity) and (Extensionality), and thus:

Proposition 5.2. *The incarceration of a retraction operator is a withdrawal operator.*

[10] We are grateful to David Makinson for suggesting this terminology.

PROOF. Let $\Bumpeq$ be a given retraction operator and $\bumpeq$ its incarceration. To show (Closure) holds, we know $K \Bumpeq \phi$ is deductively closed by (Closure) for $\Bumpeq$, as is K. Since the intersection of two deductively closed sets is deductively closed, we get $K \bumpeq \phi = Cn(K \bumpeq \phi)$. (Vacuity) follows from the fact that $\Bumpeq$ satisfies (Weak Vacuity 2) (see the end of this section). The other two postulates are proved easily using the fact that $\Bumpeq$ already satisfies the same postulate. ■

What about our subclasses of liberation operators? What happens, for instance, when we take the incarceration of a linear liberation operator? Suppose $\Bumpeq$ is a linear liberation operator. Then by definition $\Bumpeq = \Bumpeq_\rho$ for some K-sequence ρ. Now modify ρ to get a new sequence $f(\rho)$ as follows. Given $\rho = (\beta_1, \ldots, \beta_m)$ we just replace each β_i by $\beta_i \vee \beta_1$ (for $i > 1$), i.e., we define

$$f(\rho) = (\beta_1, (\beta_2 \vee \beta_1), (\beta_3 \vee \beta_1), \ldots, (\beta_n \vee \beta_1))$$

Since β_1 is unchanged, $f(\rho)$ is again a K-sequence. Furthermore,

PROPOSITION 5.3. *Let ρ be a K-sequence and $\bumpeq$ be the incarceration of $\Bumpeq_\rho$. Then $\bumpeq = \Bumpeq_{f(\rho)}$.*

PROOF. Let $\rho = (\beta_1, \ldots, \beta_m)$ be a given K-sequence. We must show that for all $\phi \in L$ we have $K \cap (K \Bumpeq_\rho \phi) = K \Bumpeq_{f(\rho)} \phi$. Firstly if $\models \phi$ then $K \cap (K \Bumpeq_\rho \phi) = K = K \Bumpeq_{f(\rho)} \phi$ as required. If $\not\models \phi$ but $\bigvee_k \beta_k \models \phi$ then $K \cap (K \Bumpeq_\rho \phi) = K \cap Cn(\emptyset) = Cn(\emptyset)$ while, since $\bigvee_k \beta_k \models \phi$ is equivalent to $\beta_1 \vee \bigvee_{k>1}(\beta_k \vee \beta_1) \models \phi$, we get $K \Bumpeq_{f(\rho)} \phi = Cn(\emptyset)$. Thus in this case too we get the required conclusion. So suppose $\bigvee_k \beta_k \not\models \phi$, equivalently, $\beta_1 \vee \bigvee_{k>1}(\beta_k \vee \beta_1) \not\models \phi$. Then $K \cap (K \Bumpeq_\rho \phi) = K \cap Cn(\beta_i)$, where $i = \min\{k \mid \beta_k \not\models \phi\}$. Since $K = Cn(\beta_1)$, this is in turn equal to $Cn(\beta_1) \cap Cn(\beta_i) = Cn(\beta_i \vee \beta_1)$. Meanwhile

$$K \Bumpeq_{f(\rho)} \phi = \begin{cases} Cn(\beta_1) & \text{if } \beta_1 \not\models \phi \\ Cn(\beta_j \vee \beta_1) & \text{otherwise,} \end{cases}$$

where $j = \min\{k \mid k > 1 \text{ and } \beta_j \vee \beta_1 \not\models \phi\}$. Hence if $\beta_1 \not\models \phi$ then $K \Bumpeq_{f(\rho)} \phi = Cn(\beta_1) = Cn(\beta_1 \vee \beta_1) = K \cap (K \Bumpeq_\rho \phi)$ as required. If however, $\beta_1 \models \phi$ then we may write $j = \min\{k \mid \beta_j \vee \beta_1 \not\models \phi\}$ and, since in this case $\beta_k \vee \beta_1 \models \phi$ iff $\beta_k \models \phi$ for all k, we get $j = \min\{k \mid \beta_k \not\models \phi\}$. Hence $K \Bumpeq_{f(\rho)} \phi = Cn(\beta_j \vee \beta_1) = K \cap (K \Bumpeq_\rho \phi)$ as required. ■

Thus the incarceration of a linear liberation operator is again a linear liberation operator which furthermore satisfies (Inclusion). Also, every linear liberation operator satisfying (Inclusion) arises as the incarceration of some linear liberation operator, namely itself. Note too, that the postulates for linear liberation together with (Inclusion) characterise the first special case of linear liberation (i.e., the sequences which satisfy (A)).

What happens when we take the incarceration of a σ-liberation operator? From Prop. 3.15 and Corollary 3.19 we know that $\Bumpeq$ forms a σ-liberation operator iff $\Bumpeq = \Bumpeq_\rho$ for some K-sequence ρ which satisfies (D). Thus we know from Prop. 5.3 that if $\bumpeq$ is the incarceration of a σ-liberation operator then $\bumpeq = \Bumpeq_{f(\rho)}$ for some K-sequence ρ which satisfies (D). We can show the following:

PROPOSITION 5.4. *If a K-sequence ρ satisfies (D) then so does $f(\rho)$.*

PROOF. Suppose $\rho = (\beta_1, \dots, \beta_m)$. Clearly $\beta_1 \models \beta_1 \vee \beta_j$ for all $1 < j$. So it remains to show that for all $1 < i < j$ either $\beta_1 \vee \beta_i \models \beta_1 \vee \beta_j$ or $(\beta_1 \vee \beta_i) \wedge (\beta_1 \vee \beta_j) \models \beta_1 \vee \bigvee_{1<k<i} (\beta_1 \vee \beta_k)$, equivalently, $\beta_1 \vee (\beta_i \wedge \beta_j) \models \bigvee_{k<i} \beta_k$. Since ρ satisfies (D) we know either $\beta_i \models \beta_j$ or $\beta_i \wedge \beta_j \models \bigvee_{k<i} \beta_k$. But the former implies $\beta_1 \vee \beta_i \models \beta_1 \vee \beta_j$, while the latter implies $\beta_1 \vee (\beta_i \wedge \beta_j) \models \bigvee_{k<i} \beta_k$. Hence $f(\rho)$ satisfies (D). ■

Thus the condition (D) on K-sequences remains *invariant* under the modification f. (The same cannot be said of (C).) Therefore every incarceration of a σ-liberation operator has the form $\Bumpeq_\rho$ for some ρ satisfying (D). Hence as a corollary we may state:

COROLLARY 5.5. *The incarceration of a σ-liberation operator is again a σ-liberation operator which furthermore satisfies (Inclusion). Also, every σ-liberation operator satisfying (Inclusion) arises as the incarceration of some σ-liberation operator, namely itself.*

5.2. Retraction to revision

To *revise* a belief set K by a given sentence ϕ means to modify K so that it includes ϕ, while preserving consistency. From each retraction operator $\Bumpeq$ for K we can define the revision operator $*$ for K via the Levi Identity:

(Levi Identity) $K * \phi = (K \Bumpeq \neg\phi) + \phi$.

The Levi Identity is usually employed to define a revision operator from a given *withdrawal* operator.

A central result in the AGM theory of belief change [1, 13] shows that if $\Bumpeq$ is a withdrawal operator then $*$ satisfies all the basic AGM postulates for revision [11]. The next result confirms that it is not necessary for $\stackrel{\sim}{-}$ to satisfy (Inclusion) for this result to go through.

PROPOSITION 5.6. *Let $\stackrel{\sim}{-}$ be a retraction operator for K and let $*$ be defined from $\stackrel{\sim}{-}$ via the Levi Identity. Then $*$ satisfies the basic AGM revision postulates (relative to K). Furthermore, for* every *operator $*$ for K which satisfies the basic AGM revision postulates there exists a retraction operator $\stackrel{\sim}{-}$ for K such that $*$ may be obtained from $\stackrel{\sim}{-}$ via the Levi Identity.*

PROOF. The first part of this proposition follows from the proof of the AGM result for withdrawal operators (see, e.g. [8]), and by noticing that in the only place in that proof where (Inclusion) is applied, namely in showing that the revision postulate "$K * \phi \subseteq K + \phi$" holds, it can be replaced with (Vacuity) (in fact (Weak Vacuity 1) – see below). To see this we have $K * \phi = (K \stackrel{\sim}{-} \neg\phi) + \phi \subseteq (K + \phi) + \phi = K + \phi$. The second part follows from the well-known result in AGM theory that every operator $*$ satisfying the basic AGM revision postulates may be obtained via the Levi Identity from a partial meet contraction operator for K (i.e., satisfying **(L1)**–**(L6)**). Clearly every partial meet contraction operator is a retraction operator according to our definition. ■

The above result shows us, then, that retraction operators are as suitable as withdrawal operators when using them as stepping-stones to revision. For linear liberation operators we can say more:

PROPOSITION 5.7. *Let $\stackrel{\sim}{-}$ and $*$ be as in the previous proposition. Then if $\stackrel{\sim}{-}$ additionally satisfies (Hyperregularity) then $*$ will satisfy both supplementary AGM revision postulates. Furthermore, for* every *operator $*$ for K which satisfies* all *the AGM revision postulates (basic plus supplementary) there exists a retraction operator $\stackrel{\sim}{-}$ for K satisfying (Hyperregularity) such that $*$ may be obtained from $\stackrel{\sim}{-}$ via the Levi Identity.*

PROOF. To show the first part of the proposition, first note that the two AGM supplementary revision postulates can equivalently be expressed as the single postulate:

$$\text{If } \neg\phi \notin K * \theta \text{ then } K * (\theta \wedge \phi) = (K * \theta) + \phi.$$

Now suppose $\neg\phi \notin K * \theta$, i.e., $\neg\phi \notin (K \stackrel{\sim}{-} \neg\theta) + \theta$. This implies $\theta \rightarrow \neg\phi \notin K \stackrel{\sim}{-} \neg\theta$ which, since $\models (\theta \rightarrow \neg\phi) \leftrightarrow \neg(\theta \wedge \phi)$ and $\stackrel{\sim}{-}$ satisfies **(L1)**, is

[11] [11] points out that $\stackrel{\sim}{-}$ is not required to satisfy (Closure) for this result. For the full list of (basic plus supplementary) AGM revision postulates we refer the reader to, e.g., [8, 11].

equivalent to $\neg(\theta \wedge \phi) \notin K \Bumpeq \neg\theta$. Since $K \Bumpeq \neg\theta = K \Bumpeq (\neg(\theta \wedge \phi) \wedge \neg\theta)$ (using **(L1)**), this in turn is equivalent to $\neg(\theta \wedge \phi) \notin K \Bumpeq (\neg(\theta \wedge \phi) \wedge \neg\theta)$. Applying (Hyperregularity) to this allows us to deduce $K \Bumpeq \neg(\theta\wedge\phi) = K \Bumpeq (\neg(\theta \wedge \phi) \wedge \neg\theta) = K \Bumpeq \neg\theta$. Hence: $K * (\theta \wedge \phi) = (K \Bumpeq \neg(\theta \wedge \phi)) + (\theta \wedge \phi) = (K \Bumpeq \neg\theta) + (\theta \wedge \phi) = ((K \Bumpeq \neg\theta) + \theta) + \phi = (K * \theta) + \phi$ as required.

The second part is shown by observing that every severe withdrawal operator is a retraction operator satisfying (Hyperregularity). From results in [17, Sect. 7] we know that, given any operator $*$ for K satisfying the full list of AGM revision postulates, there is a severe withdrawal operator which, when the Levi Identity is applied to it, yields $*$. ■

For a given retraction operator $\Bumpeq$ what happens if, instead of applying the Levi Identity to $\Bumpeq$, we first take its *incarceration* $\bumpeq$ and then apply the Levi Identity to $\bumpeq$? The next result shows that this has no effect on the resulting revision operator, i.e., that $\Bumpeq$ and $\bumpeq$ are *revision-equivalent* [13].

Proposition 5.8. *Let $\Bumpeq$ be a retraction operator for K and let $\bumpeq$ be the incarceration of $\Bumpeq$. Then, for all $\phi \in L$, $(K \Bumpeq \neg\phi) + \phi = (K \bumpeq \neg\phi) + \phi$.*

Proof. Since every retraction operator satisfies (Weak Vacuity 1) (see below), we have $(K \Bumpeq \neg\phi) + \phi \subseteq (K + \phi) + \phi = K + \phi$, and so $(K \Bumpeq \neg\phi) + \phi = (K + \phi) \cap ((K \Bumpeq \neg\phi) + \phi) = (K \cap (K \Bumpeq \neg\phi)) + \phi = (K \bumpeq \neg\phi) + \phi$. ■

The above result may seem surprising. Since it is perfectly possible that $K \Bumpeq \phi \supset K \bumpeq \phi$, it might be expected that revision based on $\Bumpeq$ could sometimes lead to a strictly larger belief set than revision based on just $\bumpeq$.

Overall, the results of this section have shown that it is possible to get a long way in the theory of belief change without (Inclusion). However we end this section by remarking that it is possible to get away with even less of the AGM contraction postulates, for, as a check of their proofs reveals, Propositions 5.2, 5.6, 5.7 and 5.8 do not even need the full power of (Vacuity); they can be derived using both of its following two weakenings, the first of which also doubles as another weakening of (Inclusion):

- $K \Bumpeq \phi \subseteq K + \neg\phi$ (Weak Vacuity 1)
- If $\phi \notin K$ then $K \subseteq K \Bumpeq \phi$ (Weak Vacuity 2)

6. Conclusion

We have provided a formal study of belief change operators that do not satisfy (Inclusion), to do justice to the intuition that dropping a belief may lead to the inclusion of others in the agent's corpus. We provided two models of liberation via retraction operators, *σ-liberation* and *linear liberation*, both of which utilised a finite *sequence* of sentences to guide the operation of belief removal. We showed that the class of σ-liberation operators is included in the class of linear liberation operators, and provided axiomatic characterisations for each class. We also characterised a number of subclasses of linear liberation, including severe withdrawal. Finally we showed how a given retraction operator can be transformed into either a withdrawal operator (satisfying (Inclusion)) or a *revision* operator.

7. Future Work

For future work we would like to generalise the σ-liberation model. Here, the belief sequences σ consisted of sentences which, intuitively, represented previous *revision* inputs the agent has received. Previous *retraction* steps which might have taken place are *not* represented. This means that we are restricting the domain of σ-liberation to those belief sets K which are formed by a process of revision *alone*. One natural way to record retraction steps would be to allow σ to include so-called *disbeliefs* $\overline{\gamma}$ (where $\gamma \in L$), as seen in [4], where $\overline{\gamma}$ indicates a retraction of γ. This would pave the way for a sequence-based model of *iterated* retraction [12]: when retracting ϕ we obtain a new sequence by appending $\overline{\phi}$ to the end of σ. This new sequence is then ready for the next input. We intend a full investigation of the properties of such a model. Other directions for further research are to consider more general models that do not satisfy (Vacuity) as well as (Inclusion), and also to find other sequence-based constructions which are able to model operations, such as AGM contraction and systematic withdrawal [14], that cannot be handled with our current ones. Finally, note that we have restricted ourselves to working with a *finitely generated* propositional language L. This choice brought representational advantages such as being able to identify a belief set with a single sentence. We would like to consider the general case involving a countable number of propositional variables.

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[12] A model of iterated *revision* based on sequences may be found in [12].

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